Lecture 6

Boundary Conditions (Cont'd)

$$-K_0(L)\, u_x(L,t) = H\left[u(L,t) - g(t)\right] \qquad \text{linear nonhomog.}$$

Nonlinear BCs:

$$u_x(L,t) = u^2(L,t)$$

Def a homogeneous (linear) BC is the condition satisfied by a trivial solution, ie. $u \equiv 0$.

Ex

$u_x(0,t) = 0$ homog. BC

$u(L,t) = g(t)$ nonhomog. BC

Chapter 2 Separation of Variables

Consider the linear homogeneous 1D heat equation with homogeneous BCs:

$$u_t = k u_{xx} \qquad 0 \leq x \leq L$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

$$u(x,0) = f(x)$$

Note To solve a nonhomogeneous problem, we need to learn first how to solve a homogeneous problem.

Separation of variables

Look for solutions of the form

$$u(x,t) = \phi(x)\, G(t)$$

Separation of variables, introduced by Bernoulli, allows one to reduce a PDE to ODE.

<u>**Note**</u> Separation of variables can only be used on <u>linear</u> <u>homogeneous</u> DEs with <u>linear homogeneous BCs</u>.

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt}; \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t)$$

Substitute these derivatives in the heat eq^{n} $u_t = k u_{xx}$:

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t) \quad \Big| \quad \frac{1}{k\phi(x)G(t)}$$

$$\underbrace{\frac{dG/dt}{kG(t)}}_{\text{function of } t \text{ only}} = \underbrace{\frac{d^2\phi/dx^2}{\phi(x)}}_{\text{function of } x \text{ only}} = -\lambda$$

Both sides should be equal to some constant, call it $-\lambda$.

Let's "ignore" IC for now (we will use it later).

We get two ODEs:

$$\frac{dG/dt}{k\,G(t)} = -\lambda \Rightarrow \boxed{\frac{dG}{dt} + \lambda k G(t) = 0}$$ time dependent problem for $G(t)$

$$\frac{d^2\phi/dx^2}{\phi(x)} = -\lambda \Rightarrow \boxed{\frac{d^2\phi}{dx^2} + \lambda\,\phi(x) = 0}$$ x dependent problem for $\phi(x)$

The goal is to construct a nontrivial solution (trivial solution is $u(x,t) \equiv 0$).

Boundary conditions

$$u(0,t) = 0 \Rightarrow \phi(0)\underbrace{G(t)}_{\neq 0} = 0 \Rightarrow \boxed{\phi(0) = 0}$$

$G(t) \neq 0$ since we need a nontrivial solution

$u(L,t)=0 \Rightarrow \phi(L)\underbrace{G(t)}_{\neq 0}=0 \Rightarrow \boxed{\phi(L)=0}$

<u>The time-dependent problem for $G(t)$</u>

$\frac{dG}{dt} + \lambda k\, G(t) = 0$: separable ODE

$\frac{dG}{G(t)} = -\lambda k\, dt$: separation of variables

$\ln|G(t)| = -\lambda k t + \tilde{C}$

$\boxed{G(t) = C e^{-\lambda k t}}$, C is an arbitrary const

<u>Boundary Value Problem (BVP) for $\phi(x)$</u>

$\frac{d^2\phi}{dx^2} + \lambda \phi(x) = 0$

$\phi(0)=0, \quad \phi(L)=0$

This is an <u>eigenvalue problem</u>. λ is called an <u>eigenvalue</u>

and $\phi(x) \neq 0$ is an associated <u>eigenfunction</u>.

<u>Def</u> λ is called an e'value if there exists a nontrivial function $\phi(x)$ that satisfies the above BVP.

We need to find all possible λ's for which we can find a nontrivial solution $\phi(x)$.

<u>Case I</u> <u>Eigenvalues and eigenfunctions for $\lambda > 0$</u>

Assume that $\phi(x) = e^{rx}$. Then $\phi' = re^{rx}$, $\phi'' = r^2 e^{rx}$.

$$\phi'' + \lambda\phi = 0$$

$$r^2 e^{rx} + \lambda e^{rx} = 0$$

$$\underbrace{e^{rx}}_{\neq 0}(r^2 + \lambda) = 0 \Rightarrow r^2 + \lambda = 0 \; : \; \underline{\text{characteristic eq}^{n}}$$

roots are $r = \pm i\sqrt{\lambda}$

Aside: operator approach

$D = \frac{d}{dx}$

$(D^2 + \lambda)\phi = 0$

$\pm i\sqrt{\lambda}$

$$\phi(x) = \cancelto{0}{C_1} \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

$$\phi(0) = 0 \Rightarrow C_1 \underbrace{\cos 0}_{1} + C_2 \cancelto{0}{\sin 0} = 0 \Rightarrow C_1 = 0$$

$$\phi(x) = C_2 \sin\sqrt{\lambda}x$$

$\phi(L)=0 \Rightarrow \underbrace{C_2}_{\neq 0} \sin\sqrt{\lambda}L = 0 \Rightarrow \sin\sqrt{\lambda}L=0$

for a nontrivial solution ϕ

$\sin\sqrt{\lambda}L=0 \Rightarrow \sqrt{\lambda}L = n\pi, \quad n=1,2,3,\dots$

$$\boxed{\begin{array}{ll} \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n=1,2,3,\dots & \text{e'values} \\ \phi_n(x) = \sin\frac{n\pi x}{L}, \quad n=1,2,3,\dots \ : & \text{e'functions} \end{array}}$$

(set $C_2=1$ for now)

<u>Case II</u>: $\lambda=0$

$\frac{d^2\phi}{dx^2}=0 \Rightarrow \phi(x) = \cancel{C_1}^{0} + C_2 x$

$\left.\begin{array}{l} \phi(0)=0 \Rightarrow C_1 + C_2\cdot 0 = 0 \Rightarrow C_1=0 \\ \phi(L)=0 \Rightarrow C_2 \cdot \underbrace{L}_{\neq 0} = 0 \Rightarrow C_2 = 0 \end{array}\right\} \Rightarrow \phi(x)\equiv 0$

Since $\lambda=0$ produces a trivial solution $\phi\equiv 0$, $\lambda=0$ is <u>NOT</u> an e'value.

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<u>Case III</u> : $\lambda < 0$. Let $s = -\lambda > 0$.

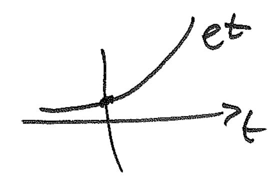
$$\frac{d^2\phi}{dx^2} - s\phi = 0$$

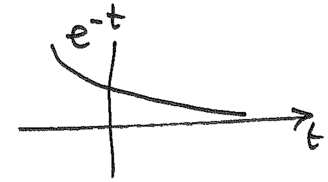
$\phi(0) = 0 \qquad \phi(L) = 0$ $\qquad\qquad$ (or $(D^2 - s)\phi = 0$)

Characteristic eqⁿ : $r^2 - s = 0$

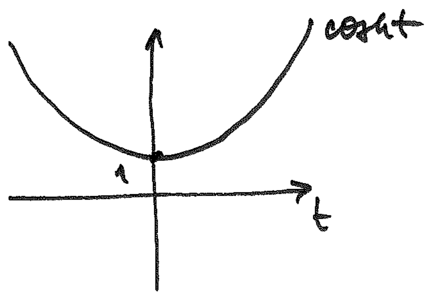
roots are $\quad r = \pm\sqrt{s}$

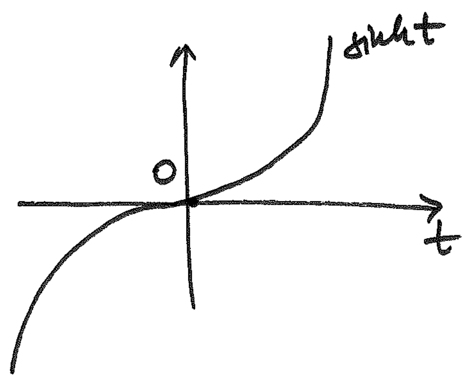
$$\phi(x) = C_1 e^{\sqrt{s}x} + C_2 e^{-\sqrt{s}x} = \boxed{C_3 \cosh\sqrt{s}x + C_4 \sinh\sqrt{s}x}$$





$$\cosh t = \frac{e^t + e^{-t}}{2} \qquad \sinh t = \frac{e^t - e^{-t}}{2}$$





$\cosh 0 = 1, \quad \sinh 0 = 0$

$\phi(x) = \cancel{c_3}^{\,0} \cosh \sqrt{s}\, x + c_4 \sinh \sqrt{s}\, x$

$\phi(0) = 0 \Rightarrow c_3 \underbrace{\cosh 0}_{1} + c_4 \cancel{\sinh 0}^{\,0} = 0 \Rightarrow c_3 = 0$

$\phi(x) = c_4 \sinh \sqrt{s}\, x$

$\phi(L) = 0 \Rightarrow c_4 \underbrace{\sinh(\sqrt{s} L)}_{\neq 0} = 0 \Rightarrow c_4 = 0$

$\Rightarrow \phi(x) \equiv 0$

There are no e'values for $\lambda < 0$, i.e. $\lambda < 0$ is not an e'value.

Summary: nontrivial $\phi(x)$ are only obtained when $\lambda > 0$:

$$\boxed{\begin{array}{lll} \lambda_n = \left(\frac{n\pi}{L}\right)^2, & n = 1, 2, 3, \ldots & \text{e'values} \\ \phi_n(x) = \sin \frac{n\pi x}{L}, & n = 1, 2, 3, \ldots & \text{e'functions} \end{array}}$$