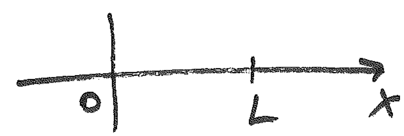
Case II : $\lambda = 0$

$$\frac{d^2\phi}{dx^2} + \cancelto{0}{\lambda}\phi = 0 \Rightarrow \frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = \cancelto{0}{c_1} + c_2 x$$

$$\phi(0) = 0 \Rightarrow \quad 0 = c_1 + c_2 \cdot 0 \Rightarrow \boxed{c_1 = 0}$$

$$\phi(x) = c_2 x$$



$$\phi(L) = 0 \Rightarrow \quad c_2 \cdot \underset{\neq 0}{L} = 0$$

$$\Rightarrow c_2 = 0$$

$$\therefore \quad \phi(x) \equiv 0$$

Hence, $\lambda = 0$ is NOT an e'value.

Case III : $\lambda < 0$. Let $s = -\lambda > 0$

$D = \frac{d}{dx}$

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \Rightarrow \frac{d^2\phi}{dx^2} - s\phi = 0$$

$(D^2 - s)\phi = 0$

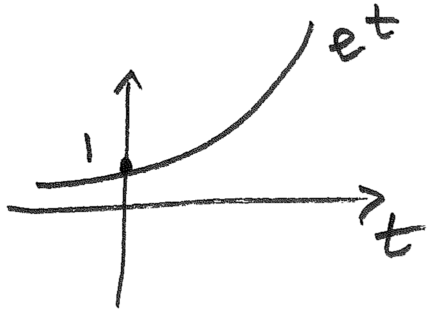
$\pm\sqrt{s}$

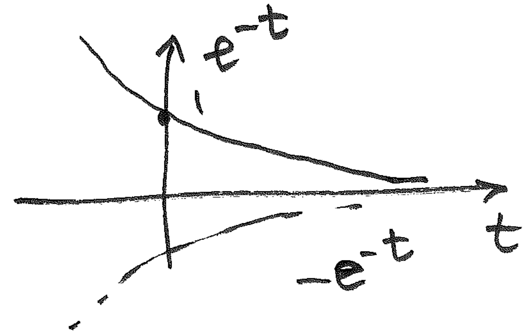
Characteristic eq^{n}: $r^2 - s = 0 \Rightarrow r = \pm\sqrt{s}$

$\phi = e^{rx}$

$$\phi(x) = C_1 e^{-\sqrt{s}x} + C_2 e^{+\sqrt{s}x} = C_3 \cosh\sqrt{s}x + C_4 \sinh\sqrt{s}x$$

Recall $\cosh t = \frac{e^t + e^{-t}}{2}$ $\sinh t = \frac{e^t - e^{-t}}{2}$





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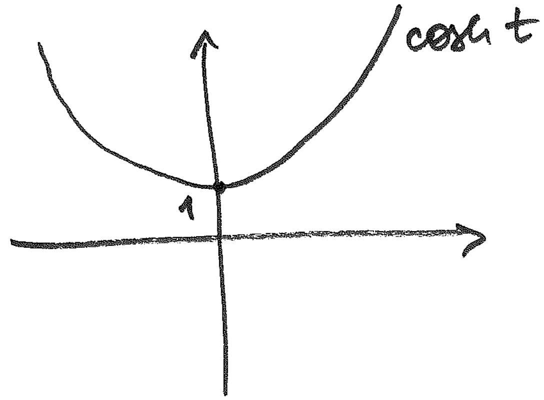

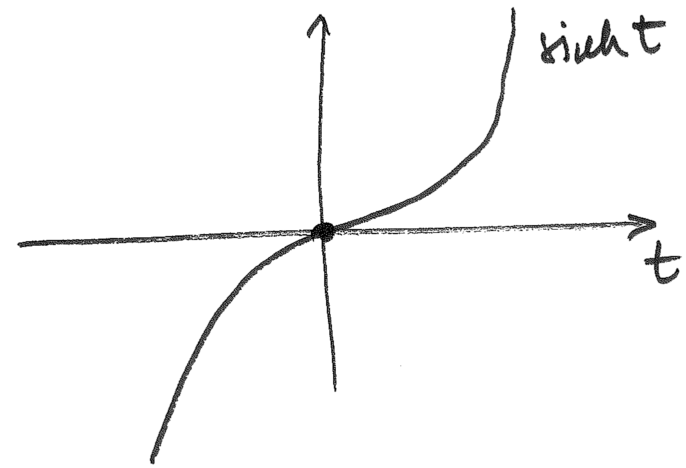

$$\phi(x) = \cancelto{0}{C_3} \cosh \sqrt{s}\,x + C_4 \sinh \sqrt{s}\,x$$

$$\phi(0) = 0 \Rightarrow \quad C_3 \underbrace{\cosh 0}_{=1} + C_4 \cancelto{0}{\sinh 0} = 0 \Rightarrow \boxed{C_3 = 0}$$

$$\phi(x) = C_4 \sinh \sqrt{s}\,x$$

$$\phi(L) = 0 \Rightarrow \quad C_4 \underbrace{\sinh(\sqrt{s}L)}_{\neq 0} = 0 \Rightarrow \boxed{C_4 = 0}$$

Hence, $\phi(x) \equiv 0 \Rightarrow$ e'values can't be negative

Hence, e'values in this problems are positive.
(nontrivial $\phi(x)$ are only available for $\lambda>0$)

Summary

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,3,\ldots$: e'values

$\phi_n(x) = \sin\frac{n\pi x}{L}$; $n=1,2,3,\ldots$: associated e'functions

Consider $\phi_1(x) = \sin\frac{\pi x}{L}$, $x \in [0,L]$

It has no roots in $(0,L)$

$\phi_2(x) = \sin\frac{2\pi x}{L}$ has 1 root in $(0,L)$

$\phi_3(x) = \frac{\sin 3\pi x}{L}$ has 2 roots in $(0,L)$

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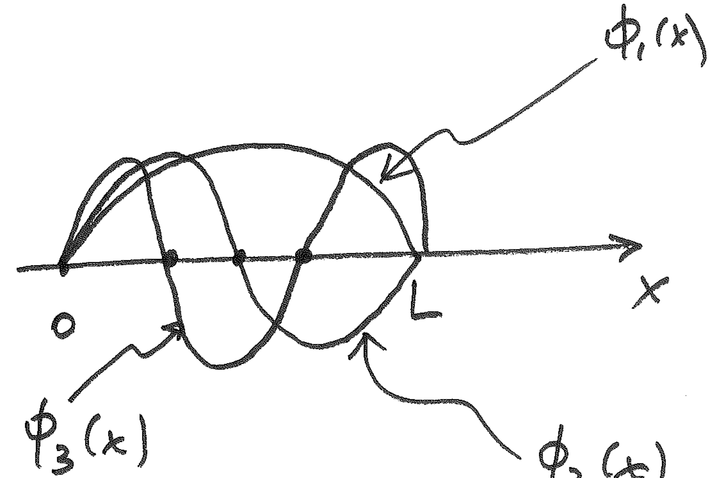

In general, one can show that $\phi_n(x)$ has exactly $n-1$ roots in $(0,L)$.

The Principle of Linear Superposition

Combining our solutions for $\phi(x)$ and $G(t)$, we can write

$$u(x,t) = \underbrace{B \sin\frac{n\pi x}{L}}_{\phi(x)} \underbrace{e^{-\left(\frac{n\pi}{L}\right)^2 k t}}_{G(t)}, \quad n = 1, 2, 3, \ldots$$

$G(t) = e^{-\lambda k t}$

$\lambda \to \lambda_n$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$

<u>Q</u> What can we say about this solution?

1. $u(x,t)$ satisfies $u_t = k u_{xx}$ (by construction)
2. $u(x,t)$ satisfies BCs $u(0,t) = u(L,t) = 0$

3. $\lim_{t \to \infty} u(x,t) = 0$ as it should be for a steady-state solution

BUT

4. $u(x,0) \neq f(x)$ in general

Superposition: since the above solution $u(x,t)$ satisfies the given linear homogeneous DE and linear homog. BCs for ALL $n = 1, 2, 3, \dots$, the linear combination of such solutions will also satisfy both DE & BCs.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

where B_n's are arbitrary constants.

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at $t=0$: $\quad u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$: <u>initial temperature</u>

we need to find B_n.

<u>Claim</u> 'Any' function (with some constraints that we will discuss later) can be written as an infinite linear combination of $\sin\frac{n\pi x}{L}$. This type of expansion is called <u>Fourier sine series</u>.

<u>Q</u> Why is it useful?

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \underbrace{e^{-k\left(\frac{n\pi}{L}\right)t}}_{\substack{\to 0 \\ \text{as } t\to\infty}} \approx \sum_{n=1}^{M} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)t}$$

for large t

Orthogonality of sines

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad \Big| \cdot \sin\frac{m\pi x}{L}$$

Q: How to find B_n's ?

Claim:

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$$

$$f(x) \cdot \sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L} \qquad \Big| \cdot \int_0^L$$

$$\int_0^L f(x) \cdot \sin\frac{m\pi x}{L}\, dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\, dx \overset{\text{swap } \int \,\&\, \sum}{=}$$

$$= \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\, dx}_{\begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}} = B_m \cdot \frac{L}{2}$$

$$\therefore \int_0^L f(x) \cdot \sin\frac{m\pi x}{L}\, dx = B_m \cdot \frac{L}{2}$$

$$\therefore B_m = \frac{2}{L} \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx$$

OR

$$\boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

<u>**Note**</u> We are allowed to switch the order of <u>infinite summation</u> and integration if the series converges <u>uniformly</u>. For now we will assume ~~that~~ this is the case.

<u>Orthogonality of functions</u>

<u>**Def**</u> Two functions $A(x)$ and $B(x)$ defined on $[0,L]$ are <u>orthogonal</u> on $[0,L]$ if

$$\int_0^L A(x)\, B(x)\, dx = 0$$