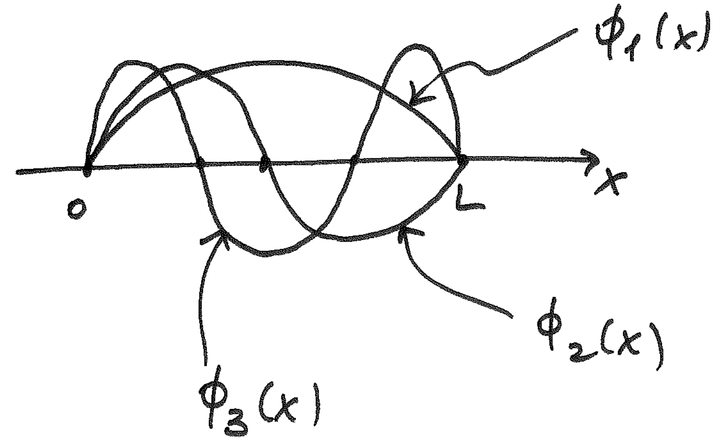
Lecture 7

Consider $\phi_1(x) = \sin\frac{\pi x}{L}$. It has no roots inside $(0, L)$.



$\phi_2(x) = \sin\frac{2\pi x}{L}$ has 1 root in $(0, L)$.

$\phi_3(x) = \sin\frac{3\pi x}{L}$ has 2 roots in $(0, L)$

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In general, one can show that $\phi_n(x)$ has $n-1$ roots in $(0, L)$.

<u>The Principle of Linear Superposition</u>

Combining our solutions for $\phi(x)$ and $G(t)$, we can write

$$u(x,t) = B \underbrace{\sin\frac{n\pi x}{L}}_{\phi(x)} \underbrace{e^{-k\left(\frac{n\pi}{L}\right)^2 t}}_{G(t)}, \qquad n = 1, 2, \ldots$$

Q What can we say about this solution?

1. $u(x,t)$ satisfies $u_t = k u_{xx}$ (by construction)

2. $u(x,t)$ satisfies BCs $u(0,t) = u(L,t) = 0$

3. $\lim_{t\to\infty} u(x,t) = 0$ as it should for a steady state solution

BUT

4. $u(x,0) \neq f(x)$ in general

<u>Superposition</u>: since the above solution $u(x,t)$ satisfies the given linear homogeneous DE and linear homogeneous BCs FOR ALL $n = 1, 2, \ldots$, the linear combination of such solutions will also satisfy both DE and BCs.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

where B_n are arbitrary constants.

at $t=0$: $\quad \underbrace{u(x,0)}_{= f(x)} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$

$f(x)$: initial temperature. We need to find B_n.

__Claim__ "Any" function (with some constraints that we will discuss later) can be written as an infinite linear combination of $\sin\frac{n\pi x}{L}$. This type of expansion is called Fourier series, specifically, Fourier sine series.

Why is it useful?

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \approx \sum_{n=1}^{M} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

for large t

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Orthogonality of sines

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \quad \Big/ \cdot \sin\frac{m\pi x}{L}$$

Q: how to find B_n's ?

Claim
$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$$

$$f(x)\sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \overset{\text{switch the order of } \int \text{ and } \sum}{=}$$

$$= \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx}_{= \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}} = B_m \cdot \frac{L}{2}$$

$$\int_0^L f(x) \sin\frac{m\pi x}{L}\, dx = \frac{L}{2} B_m$$

$$\therefore \quad \boxed{B_m = \frac{2}{L} \int_0^L f(x) \sin\frac{m\pi x}{L}\, dx}$$

Note We are allowed to switch the order of infinite summation and integration if the series converges uniformly. For now we assume that we can do this.

Orthogonality of functions

Def Two functions $A(x)$ and $B(x)$ defined on $[0, L]$ are orthogonal on $[0, L]$ if

$$\int_0^L A(x)\, B(x)\, dx = 0$$

<u>Recall</u>: vectors $\vec{A} = \langle A_1, A_2, \ldots, A_n \rangle$ and $\vec{B} = \langle B_1, B_2, \ldots, B_n \rangle$ are orthogonal if

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + \ldots + A_n B_n = \sum_{i=1}^{n} A_i B_i = 0$$

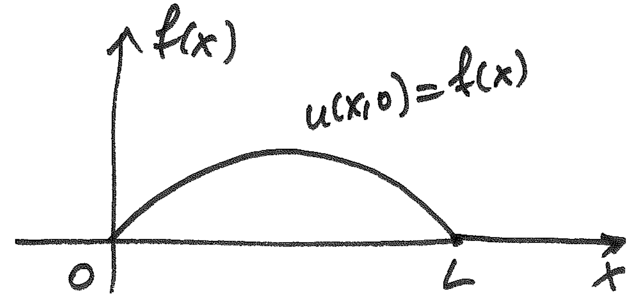
<u>Analogy</u>:

- we can think of functions as of infinite dimensional vectors
- integral $\int_0^L A(x) B(x)\, dx$ is an infinite dimensional version of the dot product $\vec{A} \cdot \vec{B}$

<u>Ex</u> $u_t = k u_{xx}$ $\quad t > 0 \quad 0 < x < L$

$u(0,t) = u(L,t) = 0$

$u(x,0) = x(L-x) = f(x)$



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We obtained

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\, f(x)\,dx$$

$$f(x) = x(L-x)$$

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\cdot x(L-x)\,dx = \left|\begin{array}{l}\text{integration} \\ \text{by parts} \\ \text{(see a handout)} \\ \text{(next page)}\end{array}\right| =$$

$$= \frac{4L^2}{(n\pi)^3}\left(1-\cos n\pi\right)$$

Here

$$\cos n\pi = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases} = (-1)^n$$

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\cdot x(L-x)\,dx = \left|\begin{array}{ll} \text{by parts} & \\ u = x(L-x) & dv = \sin\frac{n\pi x}{L}\,dx \\ du = -2x\,dx & v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L} \end{array}\right| =$$

$$= \frac{2}{L}\left\{ \cancelto{0}{x(L-x)\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L} - \frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx \right\} = \left|\begin{array}{ll} \text{by parts again} & \\ u = x & du = dx \\ dv = \cos\frac{n\pi x}{L} & \\ v = \frac{L}{n\pi}\sin\frac{n\pi x}{L} & \end{array}\right|$$

$$= -\frac{4}{n\pi}\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx \right] =$$

$$= -\frac{4}{n\pi}\left[\frac{L^2}{n\pi}\cancelto{0}{\sin n\pi} - \frac{L}{n\pi}\cdot\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L \right] = -\frac{4L^2}{(n\pi)^3}\Big(\underbrace{\cos n\pi}_{=(-1)^n} - 1\Big) =$$

$$= \frac{4L^2}{(n\pi)^3}\left(1-(-1)^n\right).$$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^3}\left[1-(-1)^n\right]\sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$