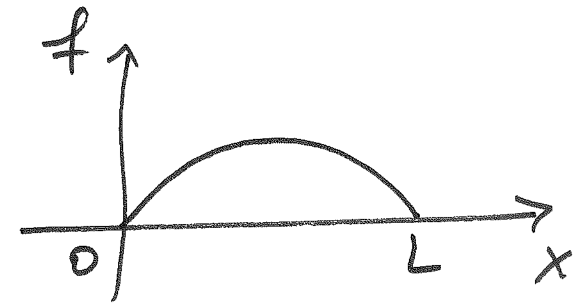
Lecture 8

Ex $\quad u_t = k u_{xx} \qquad t>0 \qquad 0<x<L$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = x(L-x) = f(x)$$



We obtained

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\, f(x)\,dx$$

$$f(x) = x(L-x)$$

$$B_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \cdot x(L-x)\, dx = \left| \begin{array}{l} \text{integration by} \\ \text{part twice} \\ \text{(see a handout)} \end{array} \right| =$$

$$= \frac{4L^2}{(n\pi)^3} \left(1 - \cos n\pi\right) = \frac{4L^2}{(n\pi)^3} \left[1 - (-1)^n\right]$$

$$\cos n\pi = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases} = (-1)^n$$

$\therefore$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^3} \left[1 - (-1)^n\right] \sin \frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\cdot x(L-x)\,dx = \left|\begin{array}{ll} \text{by parts} & \\ u = x(L-x) & dv = \sin\frac{n\pi x}{L}\,dx \\ du = -2x\,dx & v = -\frac{L}{n\pi}\cos\frac{n\pi x}{L} \end{array}\right| =$$

$$= \frac{2}{L}\left\{ \cancel{x(L-x)\left(-\frac{L}{n\pi}\right)}\cos\frac{n\pi x}{L}\Big|_0^L - \frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx \right\} = \left|\begin{array}{ll} \text{by parts again} & \\ u = x & du = dx \\ dv = \cos\frac{n\pi x}{L} & \\ v = \frac{L}{n\pi}\sin\frac{n\pi x}{L} & \end{array}\right|$$

$$= -\frac{4}{n\pi}\left[\frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx \right] =$$

$$= -\frac{4}{n\pi}\left[\frac{L^2}{n\pi}\cancel{\sin n\pi} - \frac{L}{n\pi}\cdot\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L \right] = -\frac{4L^2}{(n\pi)^3}\Big(\underbrace{\cos n\pi}_{=(-1)^n} - 1\Big) =$$

$$= \frac{4L^2}{(n\pi)^3}\left(1-(-1)^n\right).$$

Note

1. $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ decays to 0 as $t \to \infty$

2. term w/ $n=1$ decays the slowest, followed by term w/ $n=2$ etc.

3. often we will get a good approximation if we use only a finite # of terms.

4. as t increases, we can use fewer terms to approximate our solution.

Ex Take only $n=1$ term. Then

$$u(x,t) \approx \frac{4L^2}{\pi^3} \cdot 2 \sin\frac{\pi x}{L}\, e^{-k\left(\frac{\pi}{L}\right)^2 t} \equiv u_1$$

See the plot of the solution $u(x,t)$ and its approximation u_1 and the error $|u(x,t) - u_1|$

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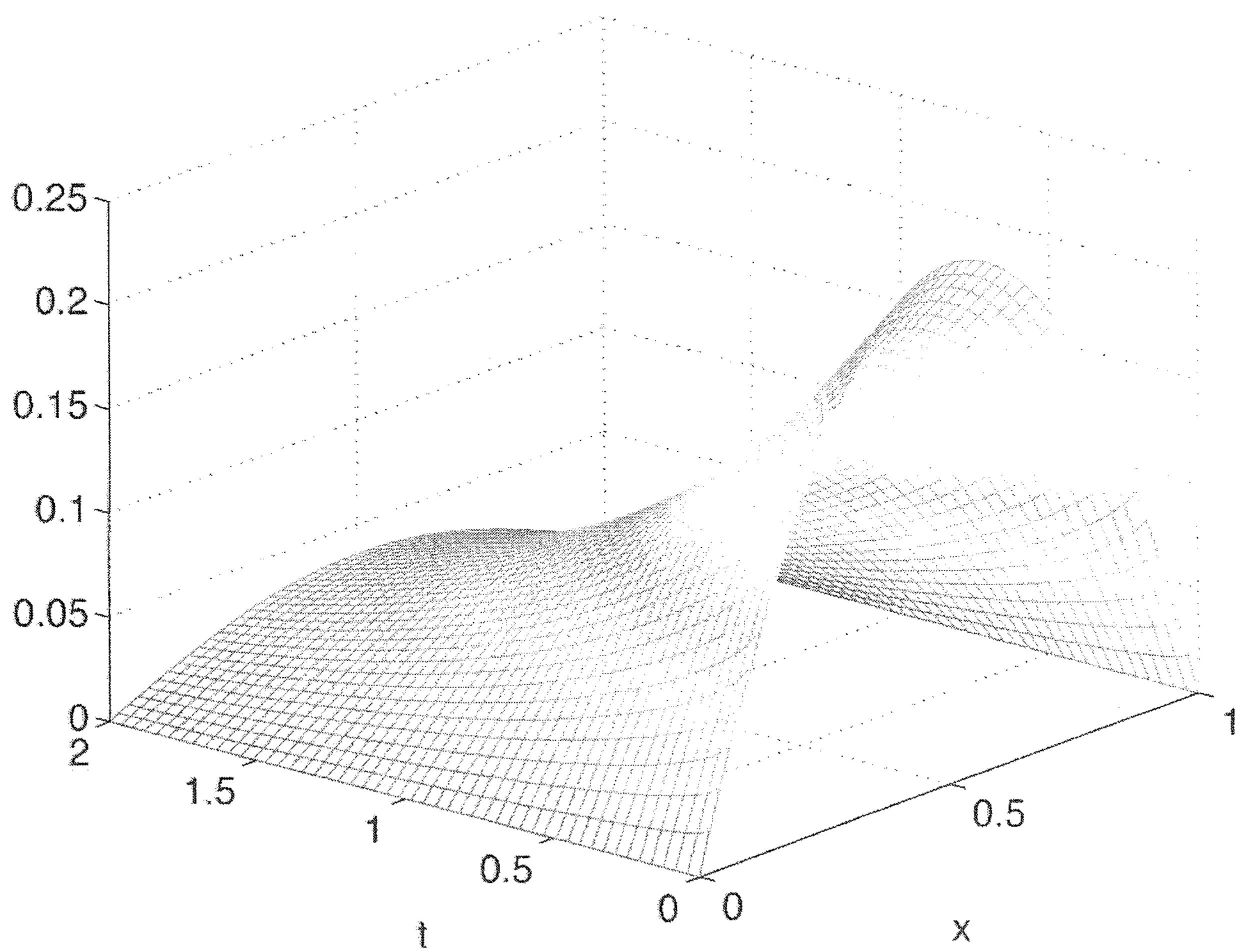
Let $L=1$ and $k=0.1$

Temperature -- u(x,t)



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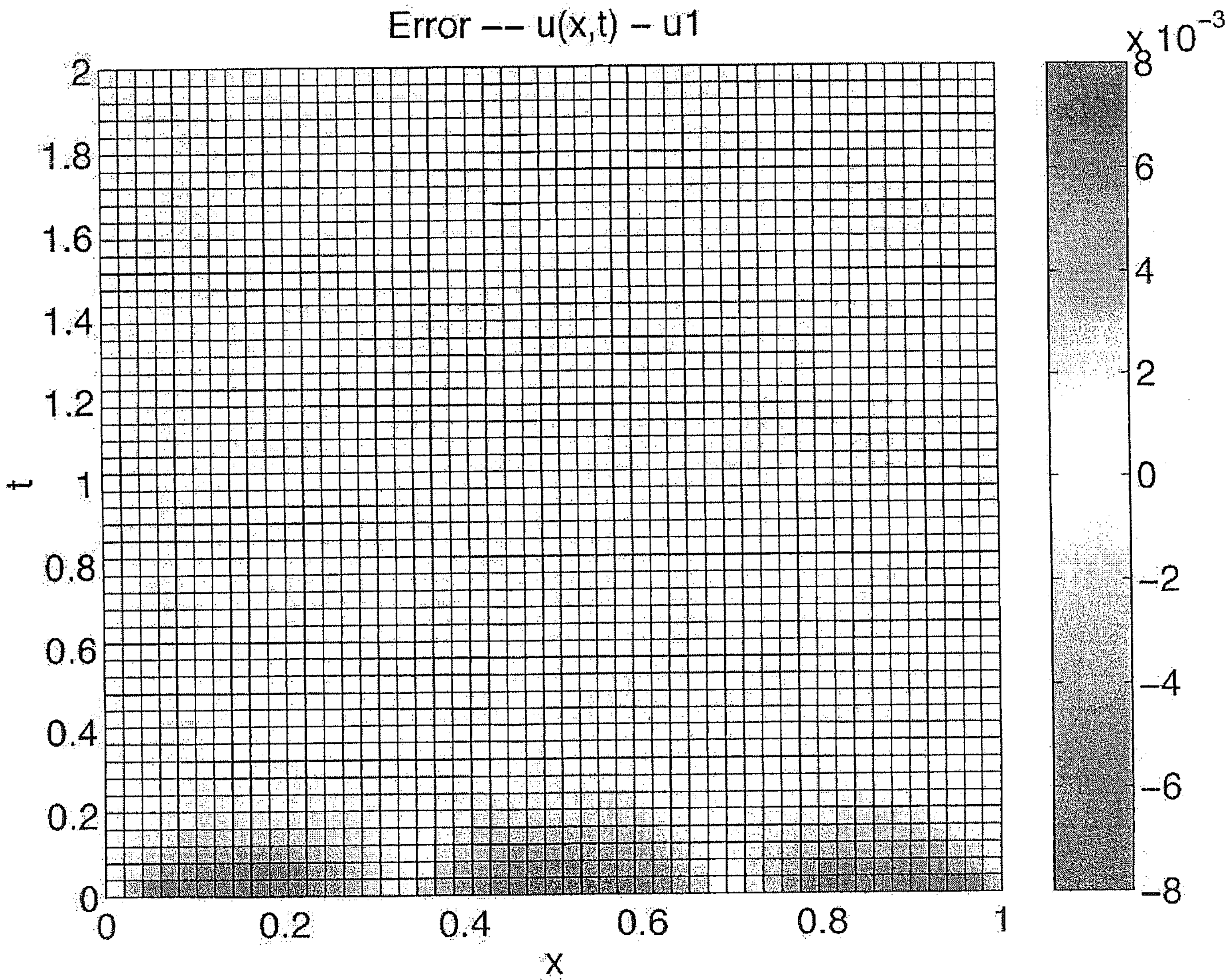
Error -- u(x,t) - u1
x 10^{-3}
8
6
4
2
0
−2
−4
−6
−8
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
t
0
0.2
0.4
0.6
0.8
1
x

Heat equation w/ homogeneous Neumann BCs

Consider

$$u_t = k\, u_{xx} \qquad 0 < x < L, \quad t > 0$$

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

$$u(x,0) = f(x)$$

Physically: perfectly insulated endpoints: no heat loss/gain at $x=0$ & $x=L$

Separation of variables

$$u(x,t) = \Phi(x) \cdot G(t)$$

$$\Rightarrow \quad \frac{dG}{dt} = -\lambda k G(t) \quad \Rightarrow \quad \boxed{G(t) = C e^{-\lambda k t}}$$

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda \phi(x) = 0 \\ \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0 \end{cases}$$

e'value problem

Case I : $\lambda > 0$

$$\frac{d^2\phi}{dx^2} + \lambda \phi = 0$$

Char. eq^{n}: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}\, i$

$$\phi(x) = C_1 \cos\sqrt{\lambda}x + \cancel{C_2 \sin\sqrt{\lambda}x}^{\,0}$$

$$\frac{d\phi}{dx} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + \cancelto{0}{C_2\sqrt{\lambda}}\cos\sqrt{\lambda}x$$

$$\left.\frac{d\phi}{dx}\right|_{x=0} = -C_1\cancelto{0}{\sqrt{\lambda}\cdot 0} + C_2\sqrt{\lambda}\cdot 1 = 0 \quad \Rightarrow \boxed{C_2 = 0}$$

$$\frac{d\phi}{dx} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x$$

$$\left.\frac{d\phi}{dx}\right|_{x=L} = -\underbrace{C_1}_{\neq 0}\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \quad \Rightarrow \quad \sin(\sqrt{\lambda}L) = 0$$

for a nontrivial sol^{n}

$$\Downarrow$$

$$\sqrt{\lambda}L = n\pi, \quad n = 1, 2, \ldots$$

$$\therefore \boxed{\begin{array}{lll} \lambda_n = \left(\frac{n\pi}{L}\right)^2, & n = 1, 2, \ldots & \text{e'values} \\ \phi_n(x) = \cos\frac{n\pi x}{L}, & n = 1, 2, \ldots & \text{e'functions} \end{array}}$$

Case II : $\lambda = 0$

$$\frac{d^2\phi}{dx^2} + \cancel{\lambda \phi}^{\,0} = 0 \Rightarrow \frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = C_1 + \cancel{C_2}^{\,0} x$$

$$\frac{d\phi}{dx} = C_2$$

$$\left.\frac{d\phi}{dx}\right|_{x=0} = 0 = C_2 \Rightarrow \boxed{C_2 = 0}$$

$$\left.\frac{d\phi}{dx}\right|_{x=L} = 0$$ gives no addition info

$\therefore$ $\phi(x) = C_1$: C_1 is an arbitrary const

if $C_1 \neq 0 \Rightarrow \phi(x) = C_1$ is a nontrivial sol^{n}

$\therefore$ $\lambda = 0$ is another e'value with associated eigenfunction $\phi(x) = 1$

Case $\overline{\text{III}}$: $\lambda < 0$: no e'values

Hence,

e'values: $\lambda_0 = 0$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

e'functions: $\phi_0(x) = 1$

$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$

OR

$$\boxed{\begin{array}{ll} \text{e'values:} & \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \ldots \\ \text{e'functions:} & \phi_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots \end{array}}$$

Solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

or

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

IC: $u(x,0) = f(x)$

$$\underset{f(x)}{\overset{\shortparallel}{u(x,0)}} = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad (1)$$

Fourier cosine series

A_n: Fourier coefficients

To find coefficients A_n, we multiply both sides of (1) by $\cos\frac{m\pi x}{L}$ and integrate from 0 to L.

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$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \int_0^L \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \overset{\text{swap}}{\underset{\int \,\&\, \sum}{=}}$$

$$= \sum_{n=0}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \;=$$

Orthogonality of Cosines

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} A_n \cdot \begin{cases} 0, & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} = A_m \cdot \begin{cases} \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

$$\therefore \int_0^L f(x) \cos \frac{m \pi x}{L} \, dx = A_m \cdot \begin{cases} \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases}$$

Then

$$\boxed{\begin{aligned} A_0 &= \frac{1}{L} \int_0^L f(x) \cancel{\cos}\, dx \\ A_m &= \frac{2}{L} \int_0^L f(x) \cos \frac{m \pi x}{L} \, dx, \qquad m \geq 1 \end{aligned}} \qquad (2)$$

Hence,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where A_0, A_n's are given above in (2).

Steady-state solution is $\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

average initial temperature