Lecture 8

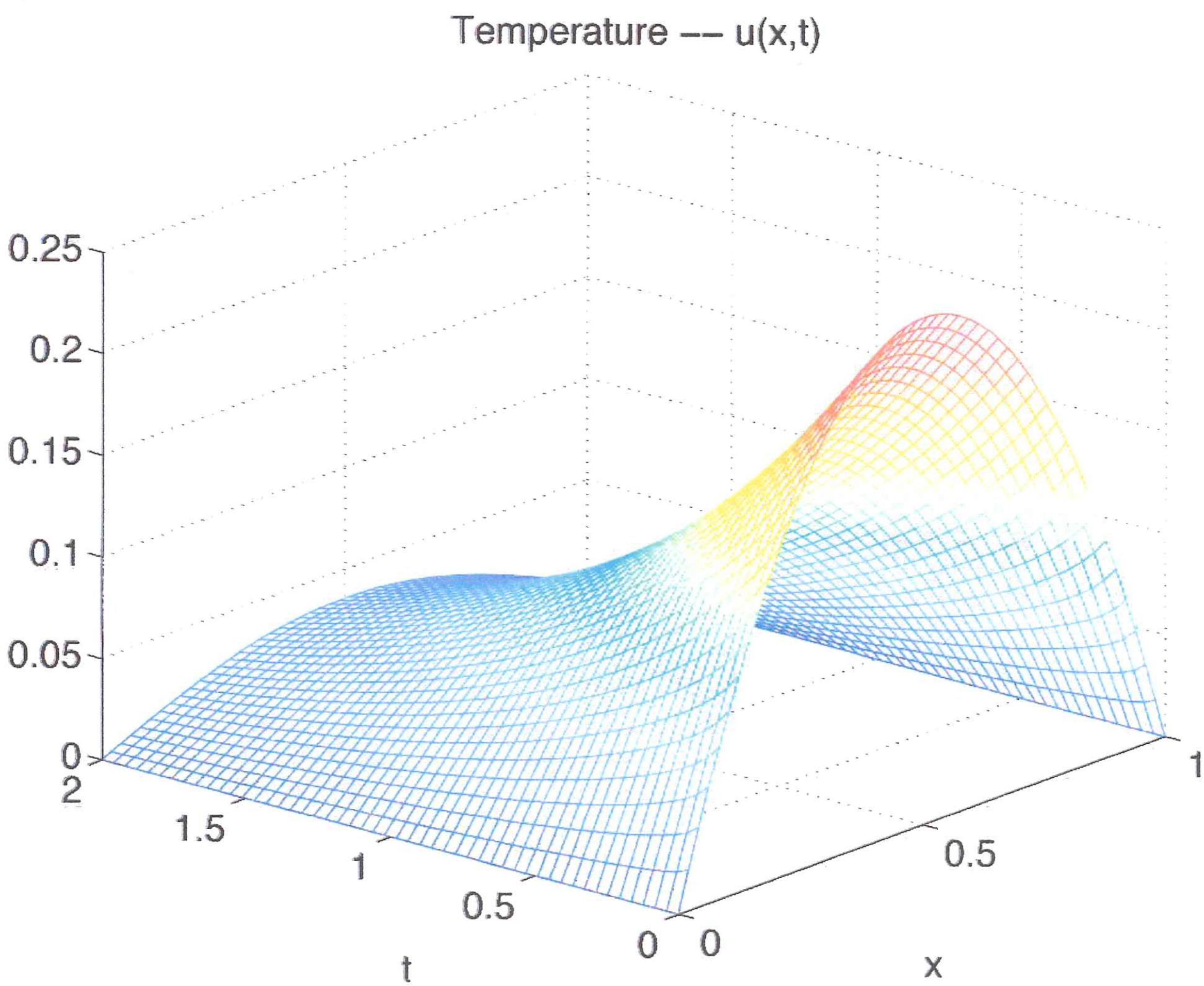
<u>Note</u>

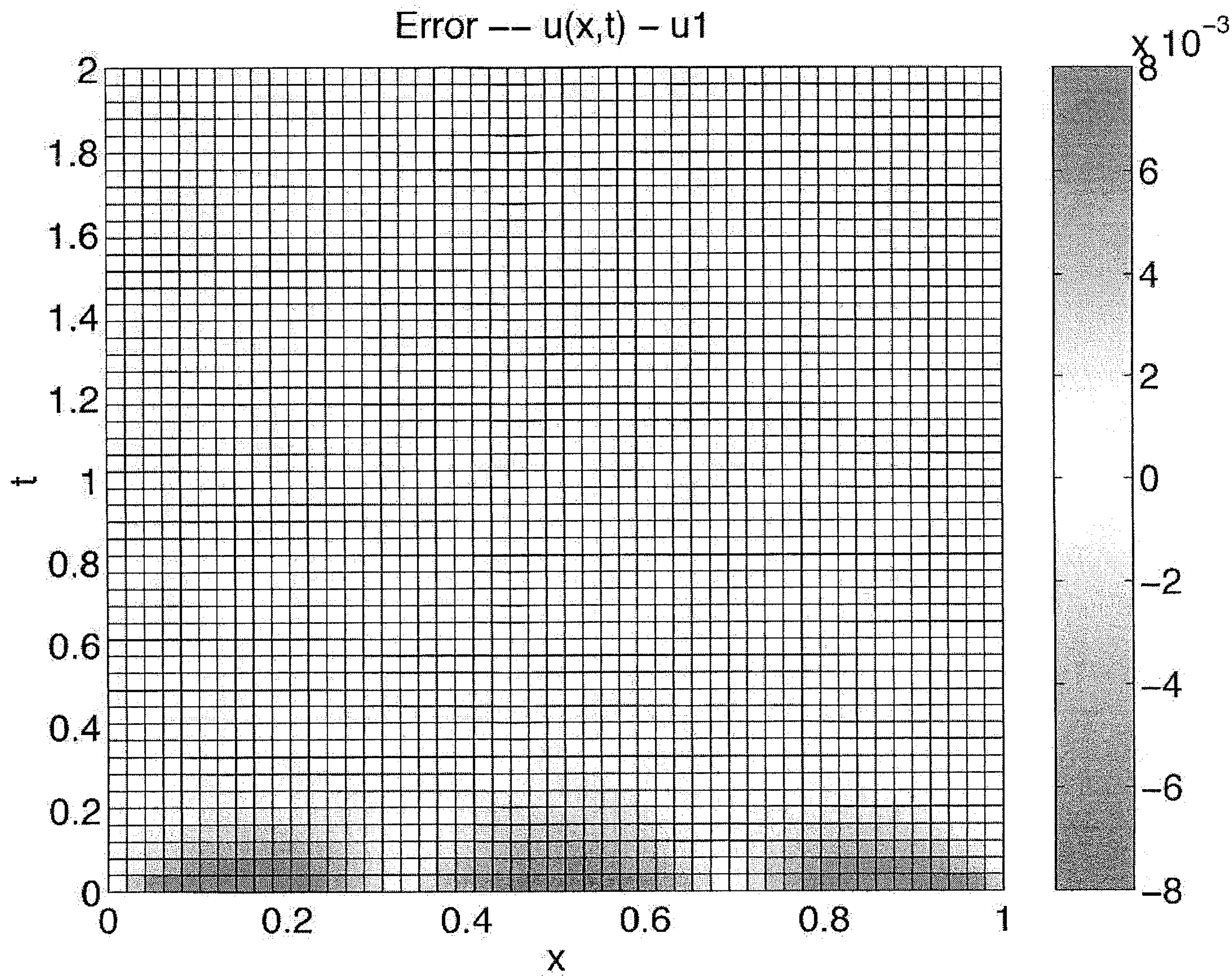
1. $e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ decays to 0 as $t \to \infty$
2. term w/ $n=1$ decays the slowest, followed by term w/ $n=2$ etc.
3. often we get a good approximation if we use only a finite # of terms
4. as t increases, we can use fewer terms to approximate our solution

<u>Ex</u> Take only $n=1$ term. Then

$$u(x,t) \approx \frac{4L^2}{\pi^3} \cdot 2 \sin\frac{\pi x}{L}\, e^{-k\left(\frac{\pi}{L}\right)^2 t}$$

Let $L=1$ and $k=0.1$. See the plot of the solution $u(x,t)$ and error.


Temperature -- u(x,t)
0.25
0.2
0.15
0.1
0.05
0
2
1.5
1
0.5
0
t
0
0.5
1
x


Error -- u(x,t) - u1
t
x
x 10^-3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
-8
-6
-4
-2
4
6
8

Heat equation w/ homogeneous Neumann BCs

Consider

$u_t = k u_{xx} \qquad 0 < x < L, \qquad t > 0$

$u_x(0,t) = 0$

$u_x(L,t) = 0$

$u(x,0) = f(x)$

Physically: perfectly insulated endpoints: no heat loss/gain at $x=0$ and $x=L$

Separation of variables

$$u(x,t) = \phi(x) \cdot G(t)$$

$$\Rightarrow \quad \frac{dG}{dt} = -\lambda k\, G(t) \quad \Rightarrow \quad \boxed{G(t) = C e^{-\lambda k t}}$$

$$\left\{ \begin{array}{l} \dfrac{d^2\phi}{dx^2} + \lambda \phi(x) = 0 \\[2ex] \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0 \end{array} \right.$$

e'value problem

<u>Case I</u> : $\lambda > 0$

$$r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm\sqrt{\lambda}\, i$$

$$\phi(x) = C_1 \cos\sqrt{\lambda}x + \cancelto{0}{C_2 \sin\sqrt{\lambda}x}$$

$$\frac{d\phi}{dx} = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + \cancelto{0}{C_2\sqrt{\lambda}\cos\sqrt{\lambda}x}$$

$$\frac{d\phi}{dx}(0) = 0 \quad \Rightarrow \quad \cancelto{0}{-C_1\sqrt{\lambda}\cdot 0} + C_2 \underbrace{\sqrt{\lambda}}_{\neq 0}\cdot 1 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$\frac{d\phi}{dx}(L)=0 \Rightarrow \underset{\substack{\neq\\0}}{-C_1}\,\underset{\substack{\neq\\0}}{\sqrt{\lambda}}\sin\sqrt{\lambda}L=0 \Rightarrow \sin\left(\sqrt{\lambda}L\right)=0$$

for a nontrivial solution

$$\sqrt{\lambda}L=n\pi,\quad n=1,2,\dots$$

$$\boxed{\begin{array}{lll} \lambda_n=\left(\frac{n\pi}{L}\right)^2, & n=1,2,\dots & \text{e'values} \\ \phi_n(x)=\cos\frac{n\pi x}{L}, & n=1,2,\dots & \text{e'functions} \end{array}}$$

Case II: $\lambda=0$

$$\frac{d^2\phi}{dx^2}=0 \Rightarrow \phi(x)=C_1+\cancel{C_2 x}^{\,0}$$

$$\frac{d\phi}{dx}=C_2$$

$\frac{d\phi}{dx}(0)=\frac{d\phi}{dx}(L)=0 \Rightarrow C_2=0$ and C_1 is an arbitrary constant

(by picking $C_1\neq 0$, we get a nontrivial solution $\phi(x)=C_1\neq 0$)

$\Rightarrow \lambda = 0$ is another e'value w/ associated e'function

$\phi(x) = 1$

Case III: $\lambda < 0$: no e'values

Hence,

e'values: $\lambda_0 = 0$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$

e'functions: $\phi_0(x) = 1$

$\phi_n(x) = \cos \frac{n\pi x}{L}, \quad n = 1, 2, \ldots$

OR

$$\boxed{\begin{array}{ll} \text{e'values:} & \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = \underline{0}, 1, 2, \ldots \\ \text{e'functions:} & \phi_n(x) = \cos \frac{n\pi x}{L}, \quad n = \underline{0}, 1, 2, \ldots \end{array}}$$

Solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

IC: $u(x,0) = f(x)$

$$\underset{u(x,0)}{\overset{\shortparallel}{f(x)}} = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \quad / \cdot \cos\frac{m\pi x}{L}$$ <u>Fourier cosine series</u>

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \int_0^L \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx =$$

$$= \sum_{n=0}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

Orthogonality of cosines

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n\neq m \\ \frac{L}{2} & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \sum_{n=0}^{\infty} A_n \underbrace{\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx}_{= \begin{cases} 0 & n\neq m \\ \frac{L}{2} & n=m\neq 0 \\ L & n=m=0 \end{cases}}$$

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = A_m \cdot \begin{cases} \frac{L}{2}, & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\boxed{\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_m &= \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx, \qquad m\neq 0,\ m\geq 1 \end{aligned}} \qquad (1)$$

Solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad =$$

where A_n's are given in (1)

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Steady-state solution is

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

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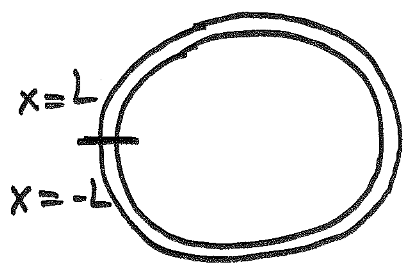
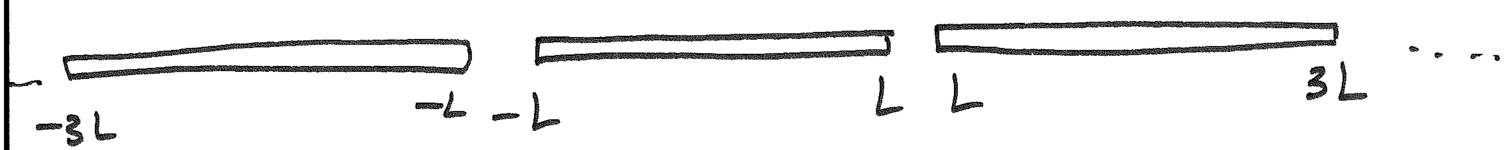
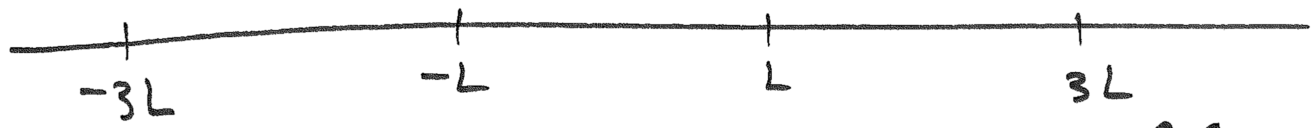
Steady-state solution is constant and depends on the initial temperature. We say that steady-state solution "has memory".

Heat Conduction in a thin ring (or Heat Conduction with periodic BCs)

Thin wire is bent into a circle. We assume that it has constant thermal properties and it is in the perfect thermal contact at the point of junction. This implies that $K_0 = \text{const}$ and temperature and heat flux are continuous at the point of junction.







mathematically we have periodic BCs.

$u_t = k\, u_{xx} \qquad -L < x < L \qquad t > 0$

$u(-L,t) = u(L,t)$ $\qquad$ $u(x,t)$ is continuous (CTS)

$u_x(-L,t) = u_x(L,t)$ $\qquad$ $u_x(x,t)$ is CTS or flux is CTS (since $K_0 = \text{const}$)

periodic BCs

$u(x,0) = f(x)$: initial condition

<u>Separation of variables</u>

Assume $\quad u(x,t) = \phi(x)\, G(t)$