Lecture 9

Heat Conduction w/ Periodic BCs

Assume $u(x,t) = \phi(x)\,G(t)$: separation of variables

$G(t) = c\,e^{-\lambda k t}$

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \\ \phi(-L) = \phi(L) \\ \dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L) \end{cases}$$

Case I : $\lambda > 0$ Char. eq^{n}: $r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}$

$e^{i\sqrt{\lambda}x},\ e^{-i\sqrt{\lambda}x}$

$\cos\sqrt{\lambda}x,\ \sin\sqrt{\lambda}x$

$\phi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$\phi'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$$\phi(-L) = \phi(L) \Rightarrow \cancel{C_1 \underbrace{\cos(\sqrt{\lambda}(-L))}_{=\cos(\sqrt{\lambda}\cdot L)}} + C_2 \underbrace{\sin(\sqrt{\lambda}(-L))}_{=-\sin(\sqrt{\lambda}L)} = \cancel{C_1 \cos(\sqrt{\lambda}L)} + C_2 \sin(\sqrt{\lambda}L)$$

$\cos(-t) = \cos t$: cosine is even f^{n}

$\sin(-t) = -\sin t$: sine is odd f^{n}

$$\Rightarrow 2C_2 \sin(\sqrt{\lambda}L) = 0 \quad \Big| \ \frac{1}{2}$$

$$\Rightarrow \boxed{C_2 \sin(\sqrt{\lambda}L) = 0} \quad (1)$$

$$\phi'(-L) = \phi'(L) \Rightarrow$$

$$-C_1\sqrt{\lambda}\underbrace{\sin(-\sqrt{\lambda}L)}_{=-\sin(\sqrt{\lambda}L)} + \cancel{C_2\sqrt{\lambda}\underbrace{\cos(-\sqrt{\lambda}L)}_{=\cos(\sqrt{\lambda}L)}} = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + \cancel{C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L)}$$

$$\Rightarrow 2C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0 \quad \Big| \ \frac{1}{2\sqrt{\lambda}} \Rightarrow \boxed{C_1 \sin(\sqrt{\lambda}L) = 0} \quad (2)$$

$\lambda > 0, \ \lambda \neq 0$

We can write equations (1), (2) as a system of two equations for unknowns C_1 and C_2:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & \sin(\sqrt{\lambda}L) \\ \sin(\sqrt{\lambda}L) & 0 \end{pmatrix}}_{A} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For a homogeneous linear system to have a nontrivial solution, i.e. $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, determinant of matrix A has to be zero. If $\det A \neq 0$, then the system has a unique solution that is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\det A = \begin{vmatrix} 0 & \sin(\sqrt{\lambda}L) \\ \sin(\sqrt{\lambda}L) & 0 \end{vmatrix} = -\sin^2(\sqrt{\lambda}L) = 0$$

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$\therefore \sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda}L = n\pi$, $n = 1, 2, \ldots$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots \; : \text{ e'values}$$

Since $\sin(\sqrt{\lambda}L) = 0$, equations (1), (2) imply that C_1, C_2 are arbitrary.

$$\therefore \phi(x) = C_1 \cos\frac{n\pi x}{L} + C_2 \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

Since C_1, C_2 are arbitrary, $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ are two e'functions associated w/ e'value $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$\phi_n(x) = \begin{cases} \cos\frac{n\pi x}{L} \\ \sin\frac{n\pi x}{L} \end{cases} \qquad n = 1, 2, \ldots$$

Case II : $\lambda = 0$

$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = c_1 + \cancel{c_2}^{\,0} x$ $\qquad \phi'(x) = c_2 = 0$

$\phi(-L) = \phi(L) \Rightarrow \cancel{c_1} - c_2 \cdot L = \cancel{c_1} + c_2 \cdot L \Rightarrow 2c_2 \cdot \underset{\neq 0}{L} = 0 \Rightarrow c_2 = 0$

$\phi(x) = c_1$

$\phi'(-L) = \phi'(L) \Rightarrow 0 = 0$ automatically satisfied

$\Rightarrow c_1$ can be arbitrary $\Rightarrow \phi(x) = c_1$ or $\phi(x) = 1$

$\Rightarrow \lambda_0 = 0$ is e'value w/ e'function $\phi_0(x) = 1$

Case III : $\lambda < 0$: no e'values

We found that

e'value: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ $\qquad \lambda_0 = 0$

e'functions: $\phi_n(x) = C_1 \cos\frac{n\pi x}{L} + C_2 \sin\frac{n\pi x}{L}$, $\qquad n = 1, 2 \ldots$

or $\phi_n(x) = \begin{cases} \cos\frac{n\pi x}{L} \\ \sin\frac{n\pi x}{L} \end{cases}$

and $\phi_0(x) = 1$ for $\lambda_0 = 0$

<u>Note</u> There is only one e'function $\phi_0(x) = 1$ associated w/ e'value $\lambda_0 = 0$ and there are two e'functions for λ_n, $n \geq 1$: $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$.

Solution (by linear superposition):

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} +$$

$$+ \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Initial condition: $u(x,0) = f(x)$

$$\underset{\substack{\parallel \\ f(x)}}{u(x,0)} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad : \quad \text{Fourier series}$$

We use orthogonality of $\left\{\cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}$ to find coefficients $a_0, a_n, b_n, n \geq 1$.

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Orthogonality Conditions

1. $$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

2. $$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

3. $$\int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0$$

Pf HW

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \Big/ \cdot \quad \cos\frac{m\pi x}{L}, \ \sin\frac{m\pi x}{L}$$

$$\int_{-L}^{L}$$

$$\int_{-L}^{L} f(x) \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx +$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx$$

includes the term w/ a_0

Using above orthogonality conditions, we get:

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx = a_m \underbrace{\int_{-L}^{L} \cos^2 \frac{m\pi x}{L} dx}_{= \begin{cases} L & m \neq 0 \\ 2L & m = 0 \end{cases}}, \qquad m \geq 0$$

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$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx = b_m \underbrace{\int_{-L}^{L} \sin^2 \frac{m\pi x}{L}\, dx}_{= L}, \quad m \geq 1$$

Hence,

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx, \quad m \geq 1 \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx, \quad m \geq 1 \end{aligned}}$$

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Summary: Boundary Value Problems for $\frac{d^2\phi}{dx^2} = -\lambda\phi$

BCs:	Dirichlet	Neumann	Periodic
	$\phi(0) = \phi(L) = 0$	$\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\psi}{dx}(L)$