MATH 480: Homework 1
Fall 2020

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page and an **assignment sheet**.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Read sections 1.1-1.5, Appendix to 1.5, sections 2.1-2.3, Appendix to 2.3, section 2.4.

1. Solve each of the following ODEs for $y(x)$:

   (a) $y'' + 16y = 0 \quad$ with $\quad y(0) = 1$ and $y'(0) = 0$.
   (b) $y'' + 6y' + 9y = 0 \quad$ with $\quad y(0) = 1$ and $y'(0) = 0$.
   (c) $y''' - y'' + y' - y = 0 \quad$ with $\quad y(0) = 1$ and $y'(0) = y''(0) = 0$.

2. Integration by parts allows us to transfer the derivative from one part of the integrand to another:
$$\int_a^b f(x)\, g'(x)\, dx = \Big[f(x)\, g(x)\Big]_a^b - \int_a^b f'(x)\, g(x)\, dx.$$
   Prove that the above statement is true. (**HINT:** think product rule!)

3. A linear PDE can be written in differential operator notation $\mathcal{L}(u) = f$, where $\mathcal{L}$ is the linear differential operator, u is the unknown function, and f is the right-hand side function. For each of the following PDEs, determine the linear operator and the right-hand side function, the order of the PDE, and whether the PDE is homogeneous or nonhomogeneous:

   (a) $u_{xxx} + u_{yyy} - u = 0$
   (b) $u_{tt} - u_{xx} + u_{yy} + u_{zz} = xyz$
   (c) $x^2 u_{xx} - y^2 u_y = \cos(x) - \sin(y)$
   (d) $y^2 u_{xx} - x^2 u_y = \cos(y) - \sin(x)$
   (e) $u_t - \cos(xt) u_{xxx} - t^5 = t^2 u$

4. The following convection-diffusion-decay equation appears in many physical applications:
$$u_t = D u_{xx} - c u_x - \lambda u\,.$$
   Show that this equation can be transformed into a heat equation for $w(x,t)$ by applying the transformation
$$u(x,t) = w(x,t) e^{\alpha x - \beta t}\,.$$

HINT: You will only obtain a heat equation for $w(x,t)$ with an appropiate choice for the constants α and β in terms of the constants D, c, and λ. Determine the choice for α and β that produces a heat equation for $w(x,t)$.

5. Consider the heat equation:

$$\begin{aligned} u_t &= \big(K_0(x)\, u_x\big)_x \\ u(0,t) &= 0 \\ u(1,t) &= 1\,, \end{aligned}$$

where $K_0(x) = e^x / \cos(x)$.

   (a) Determine the steady-state solution.

   (b) Plot the steady-state solution in MATLAB. Always clearly label all plots. Include your code.

6. Consider the heat equation:

$$\begin{aligned} u_t &= \big(K_0(x)\, u_x\big)_x + Q(t) \\ u(1,t) &= 0 \\ u(2,t) &= 1\,, \end{aligned}$$

where $K_0(x) = x^2$.

   (a) Under what condition on the heat source $Q(t)$ does a steady-state solution exist for this problem? Clearly explain your answer.

   (b) Under this condition, determine the steady-state solution.

7. Consider the function:

$$u(x,t) = \sin(4\pi x)\, e^{-\pi t}\,.$$

   (a) Plot this function in MATLAB over the domain $(x,t) \in [0,1] \times [0,1]$ using the `mesh` command. Always clearly label all plots. (**HINT:** see page 4 of the "Introduction to Plotting with MATLAB" guide available on the course webpage at: `https://www.webpages.uidaho.edu/~barannyk/Teaching/tutorial_plotting_Matlab.pdf`.)

   (b) Explain what you observe.

8. If $\mathcal{L}$ is a linear operator, prove that $\mathcal{L}\left(\sum_{n=1}^{M} c_n u_n\right) = \left(\sum_{n=1}^{M} c_n \mathcal{L} u_n\right)$.

   (*Hint:* use induction to prove this result. See Lecture 5 for an example of mathematical induction.)

9. Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

   where n, m are both integers with $n \geq 0$ and $m \geq 0$. Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right].$$

   (Be careful if $a - b = 0$ or $a + b = 0$.)

10. **Separation of Variables**. By using $u(x,t) = X(x)T(t)$ or $u(x,y,t) = X(x)Y(y)T(t)$, separate the following PDEs into two or three ODEs for X and T or X, Y, and T. The parameters c and k are constants. You do not need to solve the equations. *Note:* one of the equations cannot be separated. Indicate this when you discover that equation.

    (a) $u_{tt} = (xu_x)_x$
    (b) $u_{tt} = c^2 u_{xx}$
    (c) $u_t = k(u_{xx} + u_{yy})$
    (d) $u_t = k(yu_x + u_y)$
    (e) $u_t + cu_x = ku_{xx}$
    (f) $u_t = k(yu_x + xu_y)$

11. Consider the following boundary value problem (if necessary, see Section 2.4.1 of the textbook):

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

    (a) Give a one-sentence physical interpretation of this problem.
    (b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint*: The answer is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{-\lambda_n k t} \cos \frac{n\pi x}{L}. \Bigg].$$

    What are λ_n, A_n?

12. Consider the polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

    (a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$, and $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$.
    (b) Show that $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$.
    (c) Using the chain rule, show that $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}$ and hence $\nabla u = \hat{\mathbf{r}}\,\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial u}{\partial \theta}$.
    (d) If $\vec{A} = A_1\hat{\mathbf{r}} + A_2\hat{\boldsymbol{\theta}}$, show that $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_2)$, since $\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}$ and $\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}}$ follows from part (b).
    (e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

13. **Heat Equation with Circular Symmetry.** Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. Consider any circular annulus $a \leq r \leq b$.

    (a) Show that the total heat energy is $2\pi \int_a^b c\rho u r \, dr$.

    (b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is

    $$\phi = -2\pi b K_0 \left. \frac{\partial u}{\partial r} \right|_{r=b}.$$

    A similar results holds at $r = a$.

    (c) Assuming the thermal properties are spatially homogeneous, use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

    $$\frac{\partial u}{\partial t} = \frac{\kappa}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right).$$

    (d) Find the equilibrium temperature distribution inside the circular annulus $a \leq r \leq b$ if the outer radius is insulated and the inner radius is at temperature T.