MATH 480: Homework 2
Fall 2020

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page and an **assignment sheet (problem sheet)**.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Read section 2.5. You may need to review sections 1.5, 2.3 and 2.4.

1. If Laplace's equation is satisfied in 3D, show that

$$\oiint \nabla u \cdot \hat{\mathbf{n}}\, dS = 0\,.$$

for any closed surface. (**HINT:** use the divergence theorem.) Give a physical interpretation of this result in the context of heat flow.

2. Compute the following integrals where m and n are non-negative integers. Look out for special cases.

   (a) $\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)\, dx$

   (b) $\int_{-L}^L \cos(n\pi x/L)\cos(m\pi x/L)\, dx$

   (c) $\int_{-L}^L \cos(n\pi x/L)\sin(m\pi x/L)\, dx$

3. Consider the boundary value problem:

$$\text{PDE:} \quad u_t = u_{xx}, \ (0 < x < 4)$$

$$\text{BCs:} \quad u_x(0,t) = -2, \ u_x(4,t) = -2$$

$$\text{ICs:} \quad u(x,0) = \begin{cases} 0 & \text{if } \ 0 \le x \le 2 \\ 2x-4 & \text{if } \ 2 \le x \le 4. \end{cases}$$

   (a) Find the steady-state solution. (**NOTE:** There are Neumann BCs at each end. Generally, this would suggest that there would be *no* steady-state solution. In this case, however, the *same* value of the derivative is given at each end, meaning that heat leaves and enters at the same rate. You will find that the steady-state solution contains an arbitrary constant. So how do you choose this constant? Since the net heat flux into the interval $0 \le x \le 4$ is 0, the total heat energy must not depend on time. Choose the constant in the steady-state solution so that the total energy as $t \to \infty$ is the same as the energy at time $t = 0$.)

(b) Using separation of variables, find the solution $u(x,t)$ to this problem. In order to accomplish this do the following:

- Write $u(x,t) = w(x) + v(x,t)$, where $w(x)$ is the particular solution (also the *steady-state* solution) and $v(x,t)$ is the *transient* homogeneous solution.
- Write down the PDE and the BCs that $v(x,t)$ must satisfy.
- Write down the solution for $v(x,t)$ using the separation of variables results we obtained in class.
- Choose the arbitrary constants in $v(x,t)$ so that $u(x,t)$ satisfies the initial condition.

(c) Plot $u(x,t)$ as a function of x for several values of t in order to see how the temperature profile evolves from the initial condition towards the steady-state solution. (**NOTE:** use the `subplot` command in MATLAB in order to save paper. In each `subplot` window plot $u(x,t)$ at a given instant in time.) Submit your MATLAB code together with graphs.

4. Using separation of variables solve the following BVP:

$$\begin{aligned} \text{PDE:} \quad & u_{xx} + u_{yy} = 0 \\ \text{BCs:} \quad & u(0,y) = 0 \\ & u(L,y) = 0 \\ & u(x,0) = 0 \\ & u(x,H) = f(x). \end{aligned}$$

5. Using separation of variables, solve Laplace's equation inside a 60° wedge of radius a, subject to the BCs:

$$\begin{aligned} u(r,0) &= 0 \\ u(r,\pi/3) &= 0 \\ u(a,\theta) &= f(\theta). \end{aligned}$$

6. Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with mixed boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ t > 0 \\ \text{BCs:} \quad & u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0 \\ \text{ICs:} \quad & u(x,0) = f(x), \quad t = 0 \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = \phi(x)G(t)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$.

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

(d) Write the solution in terms of an infinite series with coefficients B_n, and derive a formula for the B_n in terms of an integral involving the intial condition $u(x,0) = f(x)$.

7. Find the solution of

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ t > 0 \\ \text{BCs:} \quad & \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0 \end{aligned}$$

for each of the following initial conditions:

$$\text{(a) } u(x,0) = 6 + 4\cos\frac{3\pi x}{L} \qquad \text{and} \qquad \text{(b) } u(x,0) = \begin{cases} 1 & \text{if} \quad 0 \le x < L/2 \\ 0 & \text{if} \quad L/2 \le x \le L\,. \end{cases}$$

8. Show that the drag force is zero for a uniform flow past a cylinder including circulation.

9. A stagnation point is a place where $\mathbf{u} = \mathbf{0}$. For what values of the circulation does a stagnation point exist on the cylinder?

10. From $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$, derive $u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$, $u_\theta = -\dfrac{\partial \psi}{\partial r}$.