MATH 480: Homework 3
Fall 2020

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page and an **assignment sheet (problem sheet)**.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

Review section 2.5. Read sections 3.1-3.5.

Properties of Laplace's Equation:

1. Using the maximum principle for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\vec{x})$, subject to $u = f(\vec{x})$ on the boundary, is unique.

Heat Equation:

2. Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial^2 u}{\partial x^2}$$

subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$, is *not* well-posed. (**HINT:** Solve the problem with separation of variables; actually the solution is the same as heat equation, but with κ replaced by $-\kappa$. Then show that if the initial conditions are changed by an arbitrarily small amount, for example,

$$f(x) \Longrightarrow f(x) + \frac{1}{m}\sin\left(\frac{m\pi x}{L}\right)$$

where m is an arbitrarily large integer, then the solution $u(x,t)$ changes by a large amount.)

Separation of Variables:

3. Consider the following homogeneous PDE and BCs:

$$u_t = u_{xx} \quad \text{in } 0 < x < 1$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0, \;\; u(1,t) + u_x(1,t) = 0$$

(a) Make the substitution $u(x,t) = \phi(x)\,G(t)$, separate variables, and find the equations for $\phi(x)$ and $G(t)$. Be sure to include the boundary conditions appropriately.

(b) Show that only $\lambda > 0$ produces non-trivial solutions. If $\lambda > 0$, find the equation satisfied by the eigenvalues. Unlike the previous examples that we have seen, you will *not* be able to solve for the eigenvalues explicitly.

(c) Write the equation that the eigenvalues satisfy as

$$F(\lambda) = 0.$$

Find approximate values for the first four eigenvalues by using the MATLAB function `fzero` to compute the roots of $F(\lambda)$. You will need to provide `fzero` with initial guesses; create a MATLAB plot of $F(\lambda)$ to get an idea where the roots are. (Make sure you turn this plot and your code in with the rest of the assignment.)

Fourier Series:

4. (a) Find the Fourier *sine* series for $f(x) = 1 - x$ defined on the interval $0 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the sine series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?

5. (a) Find the Fourier *cosine* series for $f(x) = 1-x$ defined on the interval $0 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the cosine series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?

6. (a) Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 - x^2 & \text{if } 0 < x \le 1 \end{cases}$$

   defined on the interval $-1 \le x \le 1$.

   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the Fourier series in the interval $-3 \le x \le 3$.

   (c) To what value does the series converge at $x = 0$?

7. Show that e^x is the sum of an even and an odd function.

8. (a) Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

   (b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x, & x > 0 \\ x^2, & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

9. There are some things wrong in the following demonstration. Find the mistakes and correct them.

   In this exercise we attempt to obtain the Fourier cosine series of e^x:

$$\text{(1)} \qquad e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

   Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

   the Fourier sine series of e^x. Differentiating again yields

$$\text{(2)} \qquad e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}.$$

   Since equations (1) and (2) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{array}{l} A_0 = 0 \\ A_n = 0 \end{array}\right\} \quad \text{(obviously wrong!)}.$$

   By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is,

$$A_n = \frac{2}{L} \int_0^L e^x \cos \frac{n\pi x}{L} dx.$$

10. The Fourier series of the function $f(x) = \cos(ax)$ on the interval $[-\pi, \pi]$, when a is not an integer, is given by

$$\cos(ax) = \frac{2a \sin(a\pi)}{\pi} \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2} \cos(nx) \right] \quad \text{for} \quad -\pi \le x \le \pi.$$

   (a) Differentiate both sides of this equation with respect to x, differentiating the series term by term, to find the Fourier series for $\sin(ax)$:

$$\sin(ax) = -\frac{2 \sin(a\pi)}{\pi} \left[\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 - a^2} \sin(nx) \right] \quad \text{for} \quad -\pi < x < \pi.$$

(b) Explain why this method for computing the Fourier series is valid.

(c) If you know the Fourier series for $\sin(ax)$ given in (a), why can you not differentiate it term by term with respect to x to derive the Fourier series for $\cos(ax)$.

(d) Now consider the Fourier series of $\sin(ax)$ given in (a) as known. Explain why it can be integrated term by term to get the Fourier expansion of $\cos(ax)$.

(e) Carry out this term by term integration from 0 to x, and use it to show that

$$A_0 = \frac{\sin(a\pi)}{a\pi} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 - a^2}.$$

11. Consider the function $f(x)$ defined on the interval $[0, L]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < a \\ \beta/\epsilon & \text{if } a < x < a + \epsilon \\ 0 & \text{if } a + \epsilon < x \leq L, \end{cases}$$

for some $a > 0$, $\epsilon > 0$ so that $a + \epsilon < L$.

(a) Find the Fourier sine series for $f(x)$.

(b) As ϵ decreases (keeping β constant), the region where $f(x)$ is nonzero shrinks, but the area under the curve remains constant. What happens to the Fourier sine series when we take the limit as $\epsilon \to 0$, with β constant.