MATH 480: Homework 4
Fall 2020

Review section 3.5. Read sections 3.6, 4.1-4.4.

1. Evaluate
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots$$
by evaluating the Fourier cosine series for x at $x = 0$ (see Section 3.5 on term-by-term integration of Fourier series for more information, in particular an example and equation (3.5.5) on page 124).

2. Let c_n be the coefficients of the complex Fourier series of $f(x)$. Show that if $f(x)$ is a real-valued function, then $c_{-n} = \bar{c}_n$.

Wave Equation:

3. Show that the solution to the initially unperturbed wave equation,
$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) = 0 \quad &\text{and} \quad u(L,t) = 0 \\ u(x,0) = 0 \quad &\text{and} \quad u_t(x,0) = g(x), \end{aligned}$$
is
$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$
where $G(x)$ is the odd extension of $g(x)$. (**HINT:** make use of the separation of variables solution that we computed in class.)

4. The total energy for the vibrating string problem can be written as
$$E = \text{Kinetic Energy} + \text{Potential Energy} = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$
Consider the case where $u(x,t)$ satisfies the wave equation with the boundary conditions $u(0,t) = u(L,t) = 0$.
   (a) Show that E is constant in time.
   (b) Calculate the energy in 1 mode.
   (c) Show that the total energy is the sum of the energies contained in each mode.

5. Consider a twice-differentiable function $u(x,t)$, and change variables to the *characteristic coordinates*, $\xi = x + ct$ and $\eta = x - ct$. Define $Y(\xi,\eta) = u(x,t)$ (that is, Y is the same function as u, but expressed as a function of the new coordinates). Show that
$$u_{tt} - c^2 u_{xx} = 0 \implies Y_{\xi\eta} = 0.$$
Use this result to show that any solution to the 1D wave equation can be written in the form
$$u(x,t) = F(x+ct) + G(x-ct).$$
for some functions F and G.

6. Consider the following problem with Neumann boundary conditions:
$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ &u_x(0,t) = u_x(L,t) = 0 \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned}$$
Using separation of variables to show that the solution to this problem can be written in the following form:
$$u(x,t) = \frac{1}{2}\Big(F(x-ct) + F(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi.$$
What are $F(x)$ and $G(x)$?

7. Consider the boundary value problem:
$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + ku = 0 \quad \text{in} \quad 0 < x < \pi \\ &u(0,t) = u(\pi,t) = 0 \\ &u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \end{aligned}$$
where $\delta, k > 0$ are known constants, and δ is *small.* This equation, which describes the vertical motion of a beam of length π with hinged ends, is called the *beam equation.* Use separation of variables to find the general solution of this equation. (**HINT:** you do not need to check the three cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ separately. Only one case yields nonzero solution; you can simply focus on that case.)

8. Consider vibrating strings of uniform density ρ_0 and tension T_0.
   (a) What are the natural frequencies of a vibrating string of length L fixed at both ends?
   (b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end (i.e. $\frac{\partial u}{\partial x}(H,t) = 0$)? Sketch first three modes of vibration as was done in class for case (a).
   (c) Show that the modes of vibration for the *odd* harmonics (i.e. $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.