MATH 480: Homework 5
Fall 2020

Read sections 5.1-5.8.

Sturm-Liouville problem

1. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\,\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\,\phi = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$.

2. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = \phi(b) = 0.$$

   (a) Use the result from the previous problem to put this in Sturm-Liouville form.

   (b) Using the Rayleigh quotient, show that $\lambda \geq 0$.

   (c) Solve this equation subject to the boundary conditions and determine the eigenvalues and eigenfunctions. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

   (d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

   (e) Show that the n^{th} eigenfunction has $n-1$ zeros in $1 < x < b$.

3. Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where c, ρ, K_0, and α are functions of x, subject to

$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known.

   (a) Using the Rayleigh quotient, show that the eigenvalues are positive if $\alpha < 0$.

   (b) Solve the initial value problem.

   (c) Discuss the limit as $t \to \infty$.

4. Consider the fourth-order linear differential operator:

$$\mathcal{L} = \frac{d^4}{dx^4}.$$

   (a) Show that $u\mathcal{L}(v) - v\mathcal{L}(u)$ is an exact differential.

   (b) Evaluate $\int_0^1 \left[u\mathcal{L}(v) - v\mathcal{L}(u)\right] \, dx$ in terms of the boundary data for any functions u and v.

   (c) Show that $\int_0^1 \left[u\mathcal{L}(v) - v\mathcal{L}(u)\right] \, dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = \phi(1) = 0$$
$$\phi'(0) = \phi''(1) = 0.$$

   (d) For the eigenvalue problem (using the boundary conditions from part (c))

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

   show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

   (e) Show that the eigenvalues in part (d) satisfy $\lambda \leq 0$. Is $\lambda = 0$ an eigenvalue?

5. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi = 0.$$

   subject to $\phi'(0) = \phi'(1) = 0$.

   (a) Show that $\lambda > 0$.

   (b) Use the trial function $u_T(x) = (x - x^2)^2 + \alpha$, where α is a constant, to obtain an estimate for λ_1. Find the optimal α that gives the sharpest upper bound. (**NOTE:** Be careful, remember that the trial function to estimate λ_1 must satisfy the BCs and is not allowed to have any roots in $0 < x < 1$.)

6. Determine an upper and a nonzero lower bound for the lowest frequency of vibration of a nonuniform string fixed at $x = 0$ and $x = 1$ with $c^2 = 1 + 4\alpha^2 \left(x - \frac{1}{2}\right)^2$, where α is a constant.

7. Consider the boundary value problem

$$\phi'' + \lambda\phi = 0$$
$$\phi(0) - \phi'(0) = 0$$
$$\phi(1) + \phi'(1) = 0.$$

   (a) Using the Rayleigh quotient, show that $\lambda > 0$.

(b) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

8. (See problem 3 of homework # 3). Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

with

$$\phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

9. Consider the same eigenvalue problem as in problem 8:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

with

$$\phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

(a) Determine the lowest eigenvalue using a root finding algorithm (e.g. Newton's method) on a computer. You can use the MATLAB function `fzero`.

(b) Compare your result to the bound obtained using the Rayleigh quotient in problem 8.