MATH 480: Homework 6
Fall 2020

Read sections 5.9, 5.10, 7.1-7.8

Large eigenvalues: asymptotic behavior

1. Consider the following problem

$$\frac{d}{dx}\left[(1+x)^2\frac{d\phi}{dx}\right] + \lambda(1+x)\phi = 0$$

with boundary conditions

$$\phi(0) = \phi(1) = 0.$$

Estimate (to leading order):

(a) the large eigenvalues λ_n;
(b) the corresponding eigenfunctions $\phi_n(x)$;
(c) using Matlab, plot first four asymptotic eigenfunctions.

Approximation properties

2. Consider the Fourier sine series for

$$f(x) = \begin{cases} x & \text{if} \quad x \leq 1/2 \\ 1-x & \text{if} \quad x \geq 1/2 \end{cases}$$

on the interval $0 \leq x \leq 1$.

(a) How many terms in the series should be kept so that the relative mean-square error:

$$RelError(M) = \frac{|\int_a^b f^2 \sigma dx - \sum_{n=1}^M a_n^2 \int_a^b \phi_n^2 \sigma dx|}{|\int_a^b f^2 \sigma dx|}$$

is 10^{-4}? 10^{-5}? 10^{-6}? 10^{-7}?

(b) Based on your answers for part (a), how does the relative mean-square error depend on the number of terms in the Fourier series expansion? (HINT: assume a relationship of the form $RelError \propto M^p$ and estimate p by looking at your data.)

(c) In MATLAB plot

$$\left| f(x) - \sum_{n=1}^{M} a_n \phi_n(x) \right|$$

for the four values of M found in part (a).

3. (a) Start with the Fourier sine series of $f(x) = x$ on the interval $(0, L)$. Apply Parseval's identity. Find the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   (b) Start with the Fourier cosine series of $f(x) = x^2$ on the interval $(0, L)$. Apply Parseval's identity. Find the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Higher Dimensional Equations

4. The vertical displacement of a non-uniform membrane satisfies

$$u_{tt} = c^2(u_{xx} + u_{yy}),$$

where $c = c(x, y)$. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

   (a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

   Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \quad \text{on the boundary.}$$

   What is $\sigma(x, y)$?

   (b) Prove that eigenfunctions belonging to different eigenvalues are orthogonal.

   (c) Prove that all eigenvalues are real.

   (d) Prove that $\lambda > 0$.

Gram-Schmidt Orthogonalization Method

5. Legendre polynomials $\psi_n(x)$ of degree n may be obtained by applying Gram-Smidt orthogonalization method to the linearly independent system $\{1, x, x^2, x^3, \ldots\}$ on the interval $[-1, 1]$ with weight function $\sigma = 1$. The first four are

$$\psi_0 = 1, \quad \psi_1 = x, \quad \psi_2 = x^2 - \frac{1}{3}, \quad \psi_3 = x^3 - \frac{3}{5}x$$

   (a) Find the 4th degree Legendre polynomial $\psi_4(x)$.

   (b) Express x^4 as a linear combination of the first five Legendre polynomials $\{\psi_0, \psi_1, \psi_2, \psi_3, \psi_4\}$.

   (c) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ on the interval $[0, \pi]$ to get an orthogonal pair.

6. Solve the heat equation
$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$
inside a circle of radius a with zero temperature around the entire boundary, if initially
$$u(r, \theta, 0) = 100.$$
Briefly analyze $\lim_{t\to\infty} u(r, \theta, t)$. Compare this to what you expect to occur using physical reasoning at $t \to \infty$.

7. Consider Poisson's equation on a square domain:
$$u_{xx} + u_{yy} = Q(x, y), \qquad 0 \le x \le \pi, \quad 0 \le y \le \pi$$
$$u(x, 0) = u_y(x, \pi) = 0, \quad u_x(0, y) = u(\pi, y) = 0.$$
   (a) A related problem is the Helmholtz equation on the same domain with the same BCs as above:
   $$\phi_{xx} + \phi_{yy} = -\lambda \phi.$$
   Solve this Helmholtz equation using separation of variables, i.e. find eigenvalues λ and eigenfunctions $\phi(x, y)$.

   (b) Assume that the solution to the Poisson equation can be written as
   $$u(x, y) = \sum_n \sum_m a_{nm} \phi_{nm}(x, y),$$
   where $\phi_{nm}(x, y)$ are the solutions to the Helmholtz equation from part (a). Substitute this expansion for u into the given PDE, simplify and determine an explicit formula for the coefficients a_{nm} in terms of the source term $Q(x, y)$.

   Note:

   - Assume that the order of differentiation, summation and integration can be interchanged.
   - $\displaystyle\int_0^\pi \int_0^\pi \phi_{nm}^2(x, y)\, dx\, dy = \frac{\pi^2}{4}.$

8. Consider the heat equation with a temperature dependent heat source ($Q = 4u$) in the rectangular domain $0 < x < 1$ and $0 < y < 1$:
$$u_t = u_{xx} + u_{yy} + 4u$$
$$u_x(0, y, t) = 0 \quad \text{and} \quad u_x(1, y, t) = 0$$
$$u(x, 0, t) = 0 \quad \text{and} \quad u_y(x, 1, t) + u(x, 1, t) = 0.$$

(a) Solve this problem for a general initial condition

$$u(x, y, 0) = \alpha(x, y),$$

using separation of variables. You will not be able to determine explicitly the eigenvalues, but you should write down the relationship they satisfy.

(b) What happens as $t \to \infty$? Think about this carefully.

(c) Determine all of the generalized Fourier coefficients by applying the initial condition:

$$u(x, y, 0) = \begin{cases} 1 & \text{if} \quad \frac{1}{4} < x < \frac{3}{4},\ \frac{1}{4} < y < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

(d) Plot your solution in MATLAB using the `mesh` command at

$$t = 0,\ 0.01,\ 0.02,\ 0.03,\ 0.04,\ 0.05,\ 0.06,\ 0.07.$$

Sum up to $n = 75$ and $m = 75$. Use 50 grid points in each spatial direction. HINTS: Before you evaluate the solution $u(x, y, t)$, compute and store the eigenvalues λ_{nm} and the generalized Fourier coefficients A_{nm}. Use the `fzero` command in MATLAB to compute the eigenvalues (see HW # 3). If you want you can go to the class website, download the MATLAB file membrane.m (see **Handouts** section) that was used to create the plots shown in lecture, and modify this file for the current problem. As an initial guess in using `fzero` to compute μ_m, you can use $m\pi - \pi/4$. The files are also available in Canvas.

(e) In words, explain what you observe in your MATLAB plots. Is what you observe in the final plot ($t = 0.07$) the steady-state? If not, what is it?

9. Consider the circularly symmetric wave equation with damping

$$u_{tt} + 2bu_t = \frac{1}{r}(ru_r)_r, \quad b > 0,$$

inside a circle of radius a with zero displacement around the entire boundary, i.e.

$$u(a, t) = 0.$$

Assume b is small. Initially

$$u(r, 0) = 0 \quad \text{and} \quad u_t(r, 0) = v_0.$$

(a) Using the separation of variables, solve for $u(r, t)$.

(b) Briefly analyze $\lim_{t\to\infty} u(r, t)$. Compare this to what you expect to occur using physical reasoning at $t \to \infty$.