

COROLLARY: If $\underline{L} = \underline{L}^*$, i.e. if the operator L or BVP is self-adjoint, then the Green's function is symmetric $G(x, \xi) = G(\xi, x)$.

The converse is also true.

Proof.

$$\underline{L} = \underline{L}^* \Rightarrow L = L^* \text{ and } B_i = B_i^*, \quad i=1, \dots, n.$$

If $\underline{L} = \underline{L}^*$, then from the def. of G and G^* we see that (since G is unique when it exists) $G = G^*$, i.e. $G^*(x, \xi) = G(x, \xi)$.

But from previous result, $G^*(x, \xi) = G(\xi, x)$.
Hence, $G(x, \xi) = G(\xi, x)$.

□

3.5 SOLUTION OF THE BVP (3.1) WITH INHOMOGENEOUS BCs

BVP (3.1) is

$$(3.1) \quad \left\{ \begin{array}{ll} Lu = f & x \in (a, b) \\ B_i u = g_i & i = 1, \dots, n \end{array} \right. \quad (i)$$

$$\langle \cdot, \cdot \rangle = \int_a^b \cdot \cdot \cdot dx$$

Recall that $G^*(x, \xi)$ is a solution of

$$L^* G^*(x, \xi) = \delta(x - \xi) \quad x, \xi \in (a, b) \quad (ii)$$

$$B_i^* G = 0 \quad i = 1, \dots, n$$

where

$$G^*(x, \xi) = G(\xi, x)$$

or (i)

Put solution $u(x)$ of BVP (3.1) and
 $v = G^*(x, \xi)$ of (ii) in Lagrange's
 identity

$$\langle v, Lu \rangle = [J(u, v)]_a^b + \langle L^* v, u \rangle$$

$$G(\xi, x)$$

$$\langle G^*(x, \xi), Lu \rangle = [J(u(x), G^*(x, \xi))]_{x=a}^{x=b} +$$

$$\int_a^b G(\xi, x) f(x) dx$$

$\circlearrowleft x$
indicates that we take derivatives
wrt x

$$+ \underbrace{\langle L^* G^*(x, \xi), u(x) \rangle}_{\delta(x-\xi)}$$

$$\langle G(\xi, x), f(x) \rangle = \int_a^b G(\xi, x) f(x) dx$$

$$\langle L^* G^*(x, \xi), u(x) \rangle = \langle \delta(x-\xi), u(x) \rangle =$$

$$= \int_a^b \delta(x-\xi) u(x) dx = u(\xi)$$

Relabel or interchange $x \leftrightarrow \xi$. Then

(3.14)

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi - [J_{\xi} (u(\xi), G(x, \xi))]_{\xi=a}^{\xi=b}$$

derivatives wrt ξ

Here $\left[J_{\xi}^n(u(\xi), G(x, \xi)) \right]_{\xi=a}^{\xi=b}$ involves
 the direct \nearrow Green's function and derivatives
 wrt ξ .

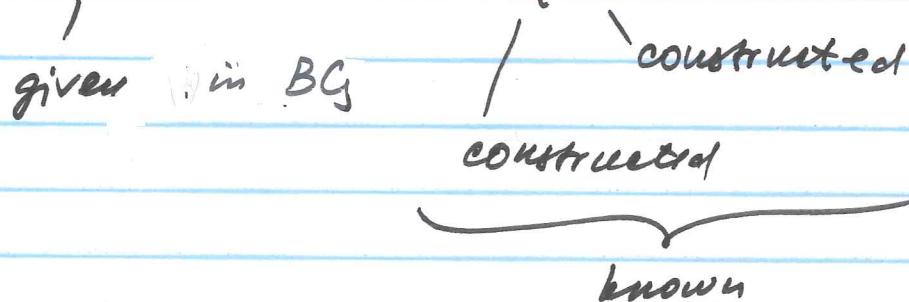
Fact (shown later for the general case):

This term can be expressed as

$$\left[J_{\xi}^n(u(\xi), G(x, \xi)) \right]_{\xi=a}^{\xi=b} = \sum_{i=1}^n B_i u \cdot B_{2n+1-i}^* G,$$

where B_s^* are known linear boundary operators, i.e. J_{ξ}^n is a linear combination

of C_i and $B_i^* G$



Ex As in §3.3 with $B_i u = C_i \neq 0$

$$(3.15) \quad L u = u'' + \lambda^2 u = f(x) \quad x \in (0, 1)$$

$$B_1 u = u(0) = C_1, \quad B_2 u = u(1) + u'(1) = C_2$$

We already have $G(x, \xi)$ provided
 $\sin \alpha + \alpha \cos \alpha \neq 0$, i.e. provided α takes
some value of α , for which G exists.

$$(3.14) \quad u(x) = \int_a^b G(x, \xi) f(\xi) d\xi - \left[J_{\xi} (u(\xi), G(x, \xi)) \right]_{\xi=a}^{\xi=b}$$

We need $J(u, v)$.

$$\langle v, Lu \rangle = \int_0^1 v(u'' + \alpha^2 u) dx \stackrel{\text{by parts twice for } u'' \text{ term}}{=} \dots$$

$$= [u'v - uv']_0^1 + \int_0^1 u(v'' + \alpha^2 v) dx$$

so,

$$J_x = v \frac{du}{dx} - u \frac{dv}{dx}$$

derivatives
wrt x

and

$$\left[J_{\xi} \underbrace{u}_{\sim} \underbrace{v}_{\sim} (u(\xi), G(x, \xi)) \right]_{\xi=0}^{\xi=1} = G(x, 1) \frac{du}{d\xi}(1) -$$

$$-u(1) \frac{\partial G(x, 1)}{\partial \xi} - G(x, 0) \frac{du}{d\xi}(0) +$$

$$+ \underbrace{u(0)}_{=c_1} \frac{\partial G(x, 0)}{\partial \xi} = \\ \text{use } B_i u = c_i$$

$$= c_1 \frac{\partial G(x, 0)}{\partial \xi} + c_2 G(x, 1) - u(1) \left[G(x, 1) + \frac{\partial G(x, 1)}{\partial \xi} \right] \\ - u'(0) G(x, 0)$$

BCs for $G(x, \xi)$:

$$B_1 G \Big|_{x=0} = G(0, \xi) = 0$$

$$B_2 G \Big|_{x=1} = G(1, \xi) + \frac{\partial G}{\partial x}(1, \xi) = 0$$

swap x and $\xi \Rightarrow$

$$G(0, x) = 0 \quad G \text{ is symmetric} \Rightarrow G(x, 0) = 0$$

$$G(1, x) + \frac{\partial G}{\partial \xi}(1, x) = 0 \quad G \text{ is symmetric}$$

$$\Rightarrow G(x, 1) + \frac{\partial G}{\partial \xi}(x, 1) = 0 \Rightarrow [..] = 0$$

Recall,

$$G(x, \xi) = \begin{cases} \frac{\sin 2x \cdot (\sin 2(\xi-1) - 2 \cos 2(\xi-1))}{2(\sin 2 + 2 \cos 2)}, & 0 \leq x < \xi \leq 1 \\ \frac{\sin 2\xi \cdot (\sin 2(x-1) - 2 \cos 2(x-1))}{2(\sin 2 + 2 \cos 2)}, & 0 \leq \xi < x \leq 1 \end{cases}$$

$$\left[\int_{\xi} (u(\xi), G(x, \xi)) \right]_{\xi=0}^{\xi=1} = c_1 \frac{\partial G(x_1)}{\partial \xi} + c_2 G(x_{11})$$

$$G(x_{11}) = - \frac{\sin 2x}{\sin 2 + 2 \cos 2}$$

$$\frac{\partial G(x_1)}{\partial \xi} = \frac{\sin 2(x-1) - 2 \cos 2(x-1)}{\sin 2 + 2 \cos 2}$$

Note: we can also verify that

$$G(x_1) = 0, \quad G(x_{11}) + \frac{\partial G}{\partial \xi}(x_{11}) = 0$$

Recall,

$$u(x) = \int_0^1 G(x, \xi) f(\xi) d\xi - \left[\int_{\xi} (u(\xi), G(x, \xi)) \right]_{\xi=0}^{\xi=1}$$

Hence,

↑ response to
f alone

$$u(x) = \int_0^x G(x, \xi) f(\xi) d\xi +$$

(3.16)

$$+ C_1 \frac{\sin 2(x-1) - 2 \cos 2(x-1)}{4\pi^2 + 2 \cos 2} + C_2 \frac{\sin 2x}{4\pi^2 + 2 \cos 2}$$

$\frac{\partial G(x_1, \xi)}{\partial \xi}$

$G(x_1, 1)$

response to C_1 alone

We have decomposition of solution to response to f alone and response to C_i alone. This is because of linearity of equation that we have sum of solutions = responses to f and C_i separately.