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Last time we found solution

$$(3.16) \quad u(x) = \underbrace{\int_0^x G(x, \xi) f(\xi) d\xi}_{\text{response to } f \text{ alone}} + C_1 \underbrace{\frac{\sin 2(x-1) - 2 \cos 2(x-1)}{\sin 2 + 2 \cos 2}}_{\text{response to } c_1 \text{ alone}} + C_2 \underbrace{\frac{\sin 2x}{\sin 2 + 2 \cos 2}}_{\text{response to } c_2 \text{ alone}}$$

We were able to decompose solution  $u(x)$  into a sum of a response due to  $f$  alone and response to  $c_i$  alone. This is possible due to linearity of the DE and BCs.

As a function of  $\lambda$ ,  $u(x)$  has simple poles at  $\sin 2 + 2 \cos 2 = 0$ . At these values  $u(x) \rightarrow \infty$ .

### Note

- (i) It is not a coincidence that the response due to  $c_i$  is a linear combination of  $u_1(x)$  and  $u_2(x)$  used to construct  $G(x, \xi)$ . For mixed BCs, we will have a linear combination of  $\tilde{u}_1(x)$  and  $\tilde{u}_2(x)$ .

(ii) If there is a parameter in the problem, which implies non-existence of  $G$  for specific values, then  $G_2(x, \xi)$  has a pole, as a function of  $\lambda$ , at these values and so does the response to  $c_i$ :

Point (i) suggests a different approach (simpler in practice) to finding the response to inhomogeneities  $c_i \neq 0$  in the boundary data.

BVP (3.1)

$$Lu = f$$

$$B_i u = c_i \quad i=1, \dots, n$$

①

This has solution  $u = u_f + u_c$  where

(i)

$$\left. \begin{array}{l} Lu_f = f \\ B_i u_f = 0 \quad i=1, \dots, n \end{array} \right\}$$

data  $\{f, \vec{\alpha}\}$

(ii)

$$\left. \begin{array}{l} Lu_c = 0 \\ B_i u_c = c_i \quad i=1, \dots, n \end{array} \right\}$$

data  $\{0, c_i\}$

This is a simple consequence of the problem being linear and is an example of

## the principle of linear superposition.

We also need a uniqueness result for the completely homogeneous problem.

$$\textcircled{2} \text{ has solutions } \Rightarrow \textcircled{1} \text{ has solution}$$

$$u_f, u_c$$

$$u = u_f + u_c$$

$$Lu = L(u_f + u_c) \stackrel{\substack{L \text{ is} \\ \text{linear}}}{=} Lu_f + Lu_c = f + 0 = f$$

$$\Rightarrow Lu = f$$

$$B_i u = B_i(u_f + u_c) \stackrel{\substack{B_i \text{ are} \\ \text{linear}}}{=} B_i u_f + B_i u_c =$$

$$= 0 + c_i = c_i$$

$$\Rightarrow B_i u = c_i$$

No non-trivial solutions (ie. no non-zero solution or only zero solution) of the completely homogeneous problem implies that the solution is unique.

This statement holds for  $\textcircled{1}$  and

$$\textcircled{2} \quad (\text{i}), (\text{ii})$$

each or ↑

① Suppose  $u$  and  $v$  are both solutions of ①.  
Consider  $w = u - v$ .

$$Lw = L(u-v) \stackrel{\substack{L \text{ is} \\ \text{linear}}}{=} Lu - Lv = f - f = 0$$

$$B_i w = B_i(u-v) \stackrel{\substack{B_i \text{ are} \\ \text{linear}}}{=} B_i u - B_i v = C_i - C_i = 0$$

$$\therefore Lw=0, B_i w=0$$

so,  $w$  satisfies completely homogeneous problem  $\Rightarrow w \equiv 0 \Rightarrow u=v \Rightarrow$  solution of ① is unique.

② (i) has solution

$$y_f(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$

$$\text{For (ii), } L u_c = 0$$

hence  $u_c$  is some linear combination of  $n$  linearly independent solutions of the homogeneous DE  $Lu=0$ , that satisfies the BCs  $B_i u_c = C_i, i=1, \dots, n$ .

In general,

$$u_c = \sum_{i=1}^n \alpha_i \tilde{u}_i(x)$$

constants  
 determined by  
 BCs  $B_i u_c = c_i$

linearly independent  
 solutions of  $L u_c = 0$

For separated/unmixed BG  $u_1(x)$  and  $u_2(x)$   
 $n=2$  are useful ( $L u_i = 0$ ;  $B_1 u_1 = 0$ ,  $B_2 u_1 \neq 0$ ;  
 $B_1 u_2 \neq 0$ ,  $B_2 u_2 = 0$ )

$$u_c = \alpha_1 u_1 + \alpha_2 u_2$$

NOT GENERAL  
 FORMULA, ONLY  
 FOR THIS CASE

$$\Rightarrow B_1 u_c = B_1 (\alpha_1 u_1 + \alpha_2 u_2) =$$

$$= \alpha_1 B_1 u_1 + \underbrace{\alpha_2 B_1 u_2}_{\neq 0} = c_1 \quad \Rightarrow \quad \alpha_2 = \frac{c_1}{B_1 u_2}$$

$$B_2 u_c = B_2 (\alpha_1 u_1 + \alpha_2 u_2) = \underbrace{\alpha_1 B_2 u_1}_{\neq 0} + \alpha_2 B_2 u_2 = c_2$$

$$\Rightarrow \alpha_1 = \frac{c_2}{B_2 u_1}$$

Then

$$u_c = \frac{c_2 u_1(x)}{B_2 u_1} + \frac{c_1 u_2(x)}{B_1 u_2}$$

homogenizing  
transformation

In example above :

$$u'' + 2^2 u = f$$

$$B_1 u = u(0) = c_1$$

$$B_2 u = u(1) + u'(1) = c_2$$

We chose

$$u_1(x) = \sin 2x$$

$$u_2(x) = \sin 2(x-1) - 2 \cos 2(x-1)$$

$$B_2 u_1 = u_1(1) + u_1'(1) = \sin 2 + 2 \cos 2$$

$$B_1 u_2 = u_2(0) = -(\sin 2 + 2 \cos 2)$$

so,

$$u_c(x) = \frac{c_2 \sin 2x}{\sin 2 + 2 \cos 2} + \frac{c_1 (\sin 2(x-1) - 2 \cos 2(x-1))}{-(\sin 2 + 2 \cos 2)}$$

or

$$u_c(x) = -C_1 \frac{\sin 2(x-1) - 2 \cos 2(x-1)}{\sin 2 + 2 \cos 2} + C_2 \frac{\sin 2x}{\sin 2 + 2 \cos 2}$$

Compare this with (3.16)