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3.6 Boundary Value Problems of Order n

$$L = a_0(x) \frac{d^n u}{dx^n} + \dots + a_{n-1}(x) \frac{du}{dx} + a_n(x)$$

$$a_j(x) \in C^n[a, b], \quad j=0, \dots, n$$

$$a_0(x) \neq 0$$

Consider BVP

$$Lu \equiv a_0 \frac{d^n u}{dx^n} + \dots + a_{n-1} \frac{du}{dx} + a_n u = f$$

$$a < x < b$$

w/ n BCs

$$B_1 u = \alpha_{11} u(a) + \dots + \alpha_{1n} u^{(n-1)}(a) +$$

$$+ \beta_{11} u(b) + \dots + \beta_{1n} u^{(n-1)}(b) = c_1$$

(3.1) - - - - -

$$B_n u = \alpha_{n1} u(a) + \dots + \alpha_{nn} u^{(n-1)}(a) +$$

$$+ \beta_{n1} u(b) + \dots + \beta_{nn} u^{(n-1)}(b) = c_n$$

We assume that rows

$$(d_{11}, \dots, d_{1n}, \beta_{11}, \dots, \beta_{1n}), \dots, (d_{nn}, \dots, d_{nn}, \beta_{nn}, \dots, \beta_{nn})$$

are independent, in particular, there is no row with all zero entries.

Data : $\{f, c_1, \dots, c_n\}$ or $\{f, c_i\}$

Problem with $f=0, c_1=\dots=c_n=0$ are said to be completely homogeneous.

If completely homog. problem has a nontrivial solution, then BVP (3.1) has either no solution or more than one solution.

If completely homog. problem

(3.17)

$$Lu=0 \quad a < x < b$$

$$B_1 u = \dots = B_n u = 0$$

has only trivial solution, then we can construct Green's function $G(x, \xi)$ associated with L, B_1, \dots, B_n . $G(x, \xi)$ satisfies

$$(3.18) \quad L G = \delta(x - \xi) \quad a < x, \xi < b$$

$$B_i G = 0 \quad i = 1, \dots, n$$

or equivalently

$$L G = 0 \quad \text{on } a < x < \xi, \quad \xi < x < b$$

$$B_i G = 0 \quad i = 1, \dots, n$$

(3.19) $G, \dots, G^{(n-1)}$ is continuous on $[a, b]$

and $\left[G^{(n-1)}(x, \xi) \right] \Big|_{\substack{x=\xi^+ \\ x=\xi^-}} = \frac{1}{a_0(\xi)}$

If $u, v \in C^n[a, b]$, then Green's formula

$$\int_a^b (v L u - u L^* v) dx = [J(u, v)]_a^b$$

$$\langle v, L u \rangle - \langle u, L^* v \rangle = [J(u, v)]_a^b$$

Conjugate

$$(3.20) \quad J(u, v) = \sum_{m=1}^n \sum_{j+k=m-1} (-1)^k \frac{d^k}{dx^k} (a_m v) \frac{d^{j+1} u}{dx^{j+1}}$$

Note that $[J(u, v)]_a^b$ can be written as the sum of dn terms

$$(3.21) \quad u(a) A_{2n} v + \dots + u^{(n-1)}(a) A_{n+1} v + \\ + u(b) A_n v + \dots + u^{(n-1)}(b) A_1 v$$

where

A_k are linear combinations of dn quantities $v(a), \dots, v^{(n-1)}(a), v(b), \dots, v^{(n-1)}(b)$

Given B_1, \dots, B_n , n independent boundary operators, we can write (3.21) to feature n quantities $B_1 u, \dots, B_n u$ instead of dn quantities $u(a), \dots, u^{(n-1)}(a)$. This suggests introducing n additional boundary (complementary) operators B_{n+1}, \dots, B_m , so that B_1, \dots, B_m are independent boundary operators.

Then (3.21) can be written as

$$(3.22) \quad [J(u, v)]_a^b = (B_1 u)(B_{2n}^* v) + \dots + (B_n u)(B_{n+1}^* v) + \\ \dots + (B_{n+1} u)(B_n^* v) + \dots + (B_m u)(B_1^* v)$$

$$(3.23) \quad [J(u, v)]_a^b = \sum_{j=1}^{2n} (B_j u) \cdot (B_{2n+1-j}^* v)$$

There are different ways to introduce B_{n+1}, \dots, B_{2n} , but they are equivalent for our purposes.

We regard B_{n+1}, \dots, B_{2n} fixed \Rightarrow by (3.23)

B_1^*, \dots, B_n^* are completely determined by B_1, \dots, B_n .

The n boundary operators B_1^*, \dots, B_n^* are adjoint to B_1, \dots, B_n .

Given n homogeneous boundary conditions

$$B_1 u = \dots = B_n u = 0$$

from (3.22) or (3.23) we can see that

$[J(u, v)]_a^b = 0$ iff v satisfies

n homogeneous adjoint BCs

$$B_1^* v = \dots = B_n^* v = 0$$

If $L = L^*$ and $B_i = B_i^*$ meaning that the adjoint conditions define the same set of functions as $B_1 u = \dots = B_n u = 0$, then the set (L, B_1, \dots, B_n) is called self-adjoint.

$$\text{Ex} \quad L = \frac{d}{dx}$$

$$\frac{du}{dx} = f \quad a < x < b$$

$$B_1 u = u(a) + \beta u(b) = C$$

Here, $L^* = -\frac{d}{dx}$ and Green's formula for $\forall u, v \in C^1[a, b]$ becomes

$$\int_a^b \left(v \frac{du}{dx} + u \frac{dv}{dx} \right) dx = [uv]_a^b = [J(u, v)]_a^b$$

$$[J(u, v)]_a^b = u(b)v(b) - u(a)v(a) = \underbrace{u(b)v(b)}_{B_1 u = u(a) + \beta u(b)} - \underbrace{u(a)v(a)}$$

$$= \underbrace{[u(a) + \beta u(b)]}_{B_1 u} \underbrace{[-v(a)]}_{B_2^* v} + \underbrace{u(b)}_{B_2 u} \underbrace{[v(b) + \beta v(a)]}_{B_1^* v}$$

$$B_1 u$$

$$B_2^* v$$

$$B_2 u$$

$$B_1^* v$$

So, we can introduce complementary boundary operator

$$B_2 u = u(b)$$

Thus,

$$B_2^* v = -v(a), \quad B_1^* v = v(b) + \rho v(a)$$

\Rightarrow adjoint boundary condition is

$$v(b) + \rho v(a) = 0$$

Note if u satisfies $B_1 u = 0$

$$\Rightarrow [J(u, v)]_a^b = 0 \quad \text{iff} \quad B_1^* v = 0$$

3.7 FREDHOLM ALTERNATIVE THEOREM

Th For the BVP (3.1)

$$Lu = f \quad x \in (a, b)$$

$$B_i u = g^i \quad i = 1, \dots, n$$

$$\langle u, v \rangle = \int_a^b uv w dx$$

EITHER

(a) The homogeneous problem ($f = c_i = 0$) has only the trivial (zero) solution. Then, the adjoint problem ($L^*v=0$, $B_i^*v=0$) only has the trivial solution, and the solution to BVP (3.1) exists and unique.
 (There exists a $G(x, \xi)$ and the operator L is uniquely invertible)

OR

The homogeneous problem has k independent non-trivial solutions, v_1, v_2, \dots, v_k .