

### 3.7 FREDHOLM ALTERNATIVE THM (Cont'd)

OR The homogeneous problem has  $k$  independent non-trivial solutions,  $u_{0i}$ . Then the adjoint <sup>homog.</sup> problem also has  $k$  independent nontrivial solutions  $v_i$ . A solution of (3.1) exists if and only if

$$(3.24) \quad \boxed{\int_a^b v_i f w dx = [J(u, v_i)]_a^b \quad i=1, \dots, k}$$

solvability condition

and it is NOT unique, but is of the form

$$\sum_{j=1}^k \mu_j u_{0j} + u_p$$

where  $u_p$  is a particular solution of (3.1) and  $\mu_j$  are arbitrary constants.

In (3.24),

$$LHS = \int_a^b v_i f w dx = \langle v_i, f \rangle$$

and per (3.23)

$$\begin{aligned}
 [\mathcal{J}(u, v)]_a^* &= \sum_{j=1}^{2n} (B_j u) \cdot (B_{2n+1-j}^* v_i^*) = \\
 &= \sum_{j=1}^n \underbrace{(B_j u)}_{\equiv c_j} \cdot (B_{2n+1-j}^* v_i^*) + \sum_{j=n+1}^{2n} (B_j u) \cdot (B_{2n+1-j}^* v_i^*) \\
 &\quad \uparrow \qquad \qquad \qquad \downarrow \\
 &\quad B_1^* v_i^*, \dots, B_n^* v_i^* \qquad \qquad \qquad 0 \qquad \qquad 0
 \end{aligned}$$

since  $v_i$  satisfies

$$L^* v_i^* = 0, \quad B_i^* v_j^* = 0$$

Hence, RHS of (3.24) is

$$\text{RHS} = \sum_{j=1}^n c_j \cdot (B_{2n+1-j}^* v_i^*) = \text{const}$$

Note

- When the homogeneous problem has a non-trivial solution (case (8)), the "solvability condition" (3.24) is a condition on the data  $\{f, c_i\}$  with respect to the null-space ( $N_L$ ) of  $L$  which is NECESSARY and SUFFICIENT for a solution of (3.1) to exist.

Generally, the operator  $L$  is not self-adjoint,  $L \neq L^*$ , so

$$N_L = \{0, u_{0i}\} \neq \{0, v_i\} = N_{L^*}$$

However,  $N_L = N_{L^*}$  (so  $u_{0i} = v_i$ ,  $i=1, \dots, k$   
wlog)  $\Leftrightarrow L = L^*$  ( $L$  is self-adjoint)

When (3.1) has homogeneous BCs,  $c_i = 0, i=1, \dots, n$ ,  
"solvability condition" (3.24)

$$(3.24') \quad \int_a^b v_i f w dx = \sum_{j=1}^n c_j \cdot (B_{2n+1-j} v_i)$$

to

$$(3.25) \quad \boxed{\int_a^b v_i f w dx = 0 \quad i=1, \dots, k}$$

Comparison w/ linear algebra.

$$Au = f$$

$$A \in \mathbb{R}^{m \times n}, \quad u \in \mathbb{R}^n \\ f \in \mathbb{R}^m$$

Inner product is the dot product

$$\langle Au, v \rangle_m = \sum_{i=1}^m v_i \cdot \left( \sum_{j=1}^n a_{ij} u_j \right) =$$

$$(\cdots)(u) = \sum_{j=1}^n u_j \sum_{i=1}^m a_{ij} v_i \quad i \leftrightarrow j$$

$$= \sum_{i=1}^n u_i \left( \sum_{j=1}^m a_{ji} v_j \right) = \langle u, A^* v \rangle_n$$

$A^*$ : adjoint ( $n \times m$ ) matrix

$A^*$  is transpose of  $A$

if  $m=n$  and  $A=A^*$   $\Rightarrow A$  is symmetric

The following problems play a role in Fredholm Alternative Thm.

$Au=0$  (direct homog. problem)

$A^*v=0$  (adjoint homog. problem)

Thm Fredholm Alternative Thm for Linear Systems

The necessary and sufficient condition for  $Au=f$  to have solution(s) is that

Solvability condition

$$\langle f, v_i \rangle = 0 \quad i=1, \dots, k$$

for every  $v_i$  that satisfies  $A^*v=0$ .

Proof

Necessity.

Suppose  $Au=f$  is solvable, i.e.  $Au=f$  for some  $u$ .

Consider

$$\langle f, v_i \rangle = \langle Au, v_i \rangle = \langle u, A^* v_i \rangle$$

$$\Rightarrow \langle f, v_i \rangle \geq 0 \text{ for every } v_i : A^* v_i = 0.$$

Sufficiency: HW (see Stargold)

Back to BVPs

2) When the homogeneous BVP has only the zero solution (case (a)),

$$N_L = N_{L^*} = \{0\}$$

we can think of the solvability condition (3.24) as being satisfied automatically since there are no functions  $v_i$  to apply.

Ex linear algebra  $2 \times 2$  case

$$A = \begin{pmatrix} 1 & a \\ 2 & 1 \end{pmatrix}$$

$$Au = f$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$u_1 + au_2 = f_1$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$2u_1 + u_2 = f_2$$

$$A^* = A^T = \begin{pmatrix} 1 & 2 \\ a & 1 \end{pmatrix}$$

$$\det A = \det A^* = 1 - 2a \neq 0 \text{ if } a \neq \frac{1}{2}$$

$\Rightarrow$  if  $a \neq \frac{1}{2}$   $\Rightarrow Au = f$  has a unique solution

$$u = \frac{1}{2a-1} \begin{pmatrix} af_2 - f_1 \\ 2f_1 - f_2 \end{pmatrix}$$

$$u = A^{-1}f$$

$$\text{let } a = \frac{1}{2}$$

$$u_1 + \frac{1}{2}u_2 = f_1$$

$$1 \cdot 2 \Rightarrow 2u_1 + u_2 = 2f_1 \quad \left. \begin{array}{l} 2u_1 + u_2 = f_2 \\ 2u_1 + u_2 = f_2 \end{array} \right\} (*)$$

$$2u_1 + u_2 = f_2$$

For system (\*) to have a solution, we can see that  $2f_1 = f_2$  or

$$\boxed{2f_1 - f_2 = 0}$$

Now, from solvability condition:

$$A^* v = 0$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ \frac{1}{2} & \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 + 2v_2 = 0$$

$$\frac{1}{2}v_1 + v_2 = 0 \quad | \cdot 2$$

$$v_1 + 2v_2 = 0$$

$$v_1 + 2v_2 = 0$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = s \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Solvability condition:

free parameter

$$\langle f, v \rangle = 0 \quad \text{or} \quad f \perp v$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \perp s \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow$$

$$\boxed{2f_1 - f_2 = 0}$$

$$\langle f, v \rangle = f \cdot v = s(2f_1 - f_2)$$