

9/25/2017

Ex 1)

$$Lu = -u'' = f(x), \quad x \in (0, 1)$$

$$\text{BCs (i)} \quad B_1 u = u'(0) = 0$$

$$B_2 u = u'(1) = 0$$

$$\text{BCs (ii)} \quad B_1 u = u(1) - u(0) = 0$$

$$B_2 u = u'(1) - u'(0) = 0$$

For this problem, L is self-adjoint for BCs (i) and (ii). The weight $w(x) = 1$.

To check if we are in case (a) or (b) of Fredholm Alternative Thm, means we need to consider homogeneous problem

$$u'' = 0$$

which has the general solution $u(x) = Ax + B$.

$$\text{BCs (i): } u'(x) = A$$

$u'(0) = u'(1) = 0 \Rightarrow A = 0$ but B is arbitrary $\Rightarrow u = B$ is a nontrivial solution of the homog problem.

so, we are in case (b), $k=1$, $u_{01}=1$.

Adjoint problem is the same as the homog. problem since $\mathcal{L}=\mathcal{L}^*$ ($v''=0$, same BCs (i) or (ii)) $v_1=1$.

Solvability condition: the BVP has a solution \Leftrightarrow

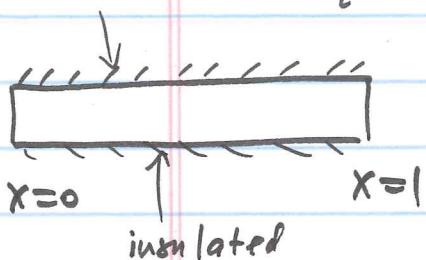
$$\int_0^1 f v_i w dx = \boxed{\int_0^1 f dx = 0}$$

$$\langle f, v_i \rangle$$

$$\left(\because v_i=1, w=1 \text{ and } B_i u=0 \right)$$

Interpretation: Heat (or diffusion) eq² with "distributed source"

insulated $u_t - u_{xx} = f(x)$ $x \in (0,1), t > 0$



rate of generation of heat internally

look for STEADY STATE solutions
 $u=u(x)$ independent of t . Then the

equation becomes

$$-u'' = f \quad x \in (0,1)$$

BCs (i) : $u'(0) = u'(1) = 0$; perfectly (thermally) insulated ends

BCs (ii) : $u(0) = u(1)$ and $u'(0) = u'(1)$

think of a thin circular ring with end joined together, also insulated at sides

$u(0) = u(1)$, $u'(0) = u'(1)$ mean that u is single-valued all around the ring (or think of periodic BCs)

Solvability condition $\int' f dx = 0$ is that the net/total generation of heat throughout the whole medium is zero. Otherwise, if this were not the case, there could not be a steady temperature profile.

Ex 2

$$Lu = u'' + \alpha^2 u = f \quad x \in (0,1)$$

$$u(x) = 1$$

$$B_1 u = u(0) = C_1$$

$$B_2 u = u(1) + u'(1) = C_2$$

(see (3.10), (3.15))

$L = L^*$ (L is self-adjoint), in other words: the homogeneous and adjoint homog. problems are the same:

For v (adjoint problem):

$$v'' + \lambda^2 v = 0$$

$$v(0) = 0, \quad v(1) + v'(1) = 0$$

Re-write homog. problem in terms of u :

$$u'' + \lambda^2 u = 0$$

$$u(0) = 0, \quad u(1) + u'(1) = 0$$

ODE has independent solutions $\sin \lambda x$ and $\cos \lambda x \Rightarrow$ general solution is

$$u(x) = A \sin \lambda x + B \overset{\circ}{\cos} \lambda x$$

$$u(0) = 0 \Rightarrow B = 0 \Rightarrow u(x) = A \sin \lambda x$$

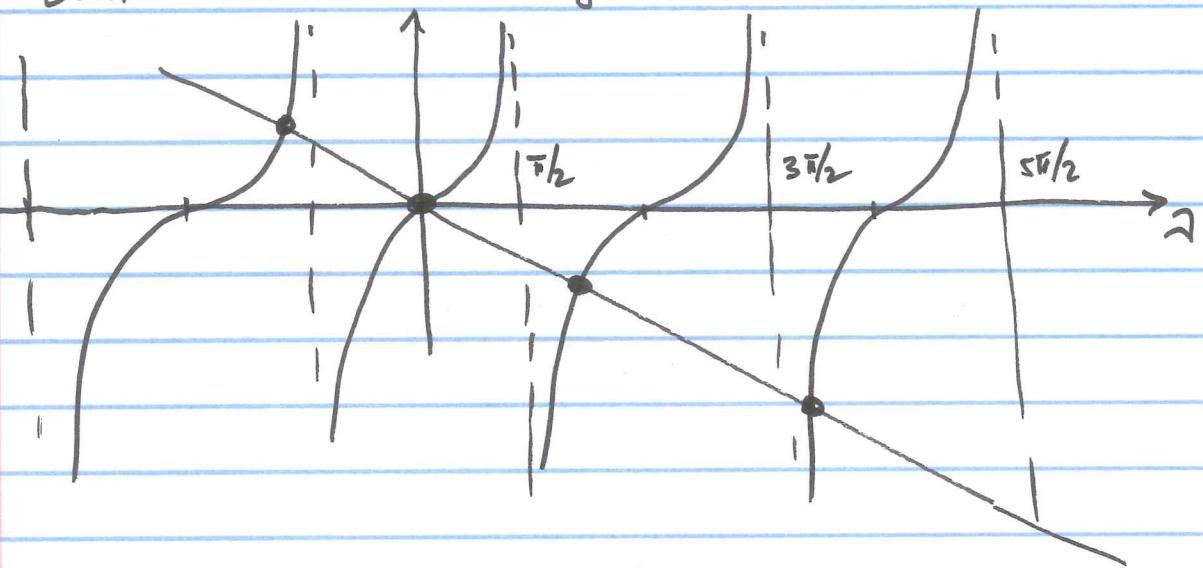
$$u'(x) = A \lambda \cos \lambda x$$

$$u(1) + u'(1) = 0 \Rightarrow A (\sin \lambda + \lambda \cos \lambda) = 0$$

$A \neq 0 \Leftrightarrow$ we have a non-trivial sol \Leftrightarrow
 $\Leftrightarrow \sin \lambda + \lambda \cos \lambda = 0$

$$\tan \gamma = -2$$

$$y = \tan^2 \gamma$$



There are ∞ many values of $\gamma : \{\gamma_i\}$ for which $\tan \gamma + 2 \cos \gamma = 0$

We may prescribe $\gamma \neq 0$. Then

$$0 < \gamma_1 < \gamma_2 < \dots$$

Then if $\gamma \neq \gamma_i$, $A=0$.

Case (a). $\exists G(x, \xi)$, we have the solution for u and it is unique (3.15).

Resonant case

$\gamma = \gamma_i$ for some i . Case (b).

In general, there is no solution for u . There is a solution (non-unique) for $u \Leftrightarrow$ solvability condition is satisfied

Case with

$$\underline{\underline{A=0}}$$

$$v''=0$$

$$v = Ax + B^0$$

$$v(0)=0 \Rightarrow B=0$$

$$v' = A$$

$$v(1) + v'(1) = 0$$

$$A + A = 0 \Rightarrow A=0$$

$$A \neq 0 : u_{0i} = \sin \pi x \quad \text{where } \tilde{\gamma} = \tilde{\gamma}_i$$

$$k=1 : v_i = \sin \pi x \quad (\text{since } L \text{ is self-adjoint})$$

Solvability condition:

$$\int_0^1 \underbrace{\sin \pi x \cdot f}_{v_i} dx = \left[J(u, \sin \pi x) \right]_0^1$$

$$\text{We found already } [J(u, v)]' = [v u' - u v']' =$$

$$v = v_i$$

$$= \underbrace{v(1)}_{\sin \pi} u'(1) - u(1) \underbrace{v'(1)}_{2 \cos \pi} - v(0) u'(0) + u(0) \underbrace{v'(0)}_{2} \quad \textcircled{E}$$

$$\text{For } v: \quad v(0) = 0$$

$$v(1) + v'(1) = 0$$

$$v = v_i = \sin \pi x \Rightarrow v_i(1) = \sin \pi$$

$$v_i' = 2 \cos \pi x$$

$$v_i'(1) = 2 \cos \pi \quad v_i'(0) = 2$$

$$\textcircled{E} \quad \sin \pi \cdot u'(1) - \underbrace{2 \cos \pi \cdot u(1)}_{=\sin \pi} + 2 u(0) \quad \textcircled{E}$$

$$\text{Note: } \sin \pi + 2 \cos \pi = 0 \Rightarrow 2 \cos \pi = -\sin \pi$$

$$\textcircled{E} \quad \sin \pi \underbrace{[u(1) + u'(1)]}_{c_2} + 2 \cdot \underbrace{u(0)}_{c_1} = c_2 \sin \pi + c_1 \cdot 2$$

i.e.

$$\int_0^1 \sin 2x \cdot f dx = C_2 \sin 2 + C_1 2$$

(3.27)

If this holds, then the solution is

$$u(x) = \mu \cdot \underbrace{\sin 2x}_{\text{arbitrary const}} + u_p$$

$$u_{01}$$

particular solution
of nonhomog. BVP (later)

Solution $u(x)$ is not unique