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(FA)

3.7 FREDHOLM ALTERNATIVE THEOREM

(Cont'd)

$$(3.1) \quad \begin{aligned} Lu &= f & x \in (a, b) \\ B_i u &= c_i \end{aligned} \quad \langle u, v \rangle = \int_a^b uv' w dx$$

Either

(a) Homog. problem for  $u$  has only zero sol<sup>y</sup>.  
Same for adjoint problem. There is a  $G(x, \xi)$ , and BVP has a unique sol<sup>y</sup>.

OR

(b) Homog. problem has  $k$  lin. independent<sup>y</sup> solutions  $u_{0i}$ ,  $i=1, \dots, k$ . Adjoint problem also has  $k$  linearly independent solutions  $v_i$ ,  $i=1, \dots, k$ .  
BVP has a solution  $\Leftrightarrow$

$$(3.24) \quad \Leftrightarrow \langle v_i, f \rangle = \left[ J(u, v_i) \right]_a^b \quad i=1, \dots, k$$

solvability conditions

(RHS depends on  $c_i$  and  $v_i$ ). This solution is not unique:

$$u = \sum_{i=1}^k \mu_i u_{0i} + u_p$$

particular solution

Note

The most direct route is to rewrite the BVP as an equivalent integral equation (Fredholm integral equation of 2<sup>nd</sup> kind) for which the result is available.

Example of proof:  $n=2$ , with unmixed BCs

- (a) Homog. problem has only zero sol<sup>n</sup>. Then Green's function  $G(x, \xi)$  exists. It was constructed in Section 3.2 - see eq<sup>n</sup> (3.4). Then

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi = u_f(x)$$

is a solution for  $B_i u = 0$ . For  $B_i u = C_i \neq 0$ , we can use (3.14):

$$(3.14) \quad u(x) = \int_a^b G(x, \xi) f(\xi) d\xi - \left[ J_{\xi} (u(\xi), G(x, \xi)) \right]_{\xi=a}^{\xi=b}$$

\ wrt to  $\xi$ , not partial derivative

OR

$$u = u_f + u_c \quad \text{with}$$

$$u_c = \frac{C_1 u_2(x)}{B_1 u_2} + \frac{C_2 u_1(x)}{B_2 u_1}$$

Uniqueness was shown in Section 3.5.

(8) Homog. problem has  $k$  lin. independent solutions  $u_{0i}, i=1, \dots, k$ .

( $\Rightarrow$ ) To show (3.24) is a NECESSARY condition  
Necessity is "easy". Use Lagrange identity where  $u$  is the solution to BVP and  $v=v_i, i=1, \dots, k$ , is a solution to the adjoint problem.

$$\langle v, Lu \rangle = \langle L^* v, u \rangle + [J(u, v)]_a^b \quad \text{Lagrange identity}$$

$v \rightarrow v_i, u$  satisfies BVP (3.1)

$$\langle v_i, Lu \rangle = \langle L^* v_i, u \rangle + [J(u, v_i)]_a^b$$

$\downarrow$   
 $\neq$                        $0$

$$\Rightarrow \langle v_i, f \rangle = [J(u, v_i)]_a^b$$

The solution is not unique since we can add arbitrary multiples of  $u_{0i}$ .

( $\Leftarrow$ ) By construction.  
Sufficiency

For a special case  $Lu \equiv -(pu')' + qu = f$

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with unmixed BCs. In this case,  $L=L^*$ ,  $w(x)=1$ .

$$Lu = -(pu')' + qu = f \quad x \in (0,1)$$

$$B_1 u = \alpha_1 u(0) + \beta_1 u'(0) = C_1$$

$$B_2 u = \alpha_2 u(1) + \beta_2 u'(1) = C_2$$

Recall,  $L=L^* \Rightarrow L=L^*$ ,  $B_i=B_i^*$ .

We showed earlier, that for any  $u, v$

$$J(u, v) = p(uv' - u'v) = p \cdot W(u, v)$$

$$\text{since } W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - v u'$$

Thm

If  $u, v$  satisfy  $Lu=0, Lv=0$ , then  
 $J(u, v) = \text{const}$

Proof

$$\langle u, L^* v \rangle = \langle L^* u, v \rangle + [J(u, v)]_a^b$$

$\uparrow$   
 $L^* u$

$\Rightarrow [J(u, v)]_a^b = 0 \Rightarrow J(u, v)$  at  $x=b$  is the same as  $J(u, v)$  at  $x=a$ .

Since  $a, b$  are arbitrary  $\Rightarrow J(u, v) = \text{const}$   $\blacksquare$

In our proof of FA, if  $u, v$  are  $\neq$  functions,  
 $L u = L v = 0 \Rightarrow J(u, v) = \text{const}$ .

We can assume wlog  $\beta_i \neq 0$ . We can write  
 $[J(u, v)]_0'$  as

$$\begin{aligned} [J(u, v)]_0' &= \frac{p(1)}{\beta_2} (u(1) B_2 v - v(1) B_2 u) - \\ (\neq) & - \frac{p(0)}{\beta_1} (u(0) B_1 v - v(0) B_1 u) \end{aligned}$$

We can show that this would be the same as  
 $[J(u, v)]_0' = [p(uv' - u'v)]_0' = p(1) (\underbrace{u(1)v'(1)} - \underbrace{u'(1)v(1)}) - p(0) (u(0)v'(0) - u'(0)v(0))$

Indeed,

$$\begin{aligned} \frac{p(1)}{\beta_2} (u(1) B_2 v - v(1) B_2 u) &= \frac{p(1)}{\beta_2} (u(1) [d_2 v(1) + \beta_2 v'(1)] - \\ & - v(1) [d_2 u(1) + \beta_2 u'(1)]) = \frac{p(1)}{\beta_2} (\beta_2 (u(1) v'(1) - v(1) u'(1))) \\ & \text{same as } \underbrace{\hspace{10em}} \end{aligned}$$

Similarly, at  $x=0$ .

Let  $\psi(x)$  be the non-zero solution of the homog. problem (so  $L\psi=0$ ,  $B_1\psi=B_2\psi=0$ ) and let  $\phi(x)$  be any other solution of  $L\phi=0$ , independent of  $\psi$  (we don't say anything about BCs for  $\phi$ ). Then

$$J(\phi, \psi) = \text{const} \stackrel{(\dagger)}{=} \frac{p(1)}{\beta_2} \left( \phi(1) B_2 \psi - \psi(1) B_2 \phi \right) - \frac{p(0)}{\beta_1} \left( \phi(0) B_1 \psi - \psi(0) B_1 \phi \right)$$

$u \rightarrow \phi$   
 $v \rightarrow \psi$

$$(**) \quad J(\phi, \psi) = \text{const} = - \frac{p(1) \psi(1) B_2 \phi}{\beta_2} + \frac{p(0) \psi(0) B_1 \phi}{\beta_1}$$

The solvability condition (3.24) is that

$$\langle \psi, f \rangle = \int_0^1 \psi(\xi) f(\xi) = [J(u, \psi)]_0^1 \stackrel{(**)}{=}$$

$$(\dagger) \quad = - \frac{p(1) \psi(1) c_2}{\beta_2} + \frac{p(0) \psi(0) c_1}{\beta_1}$$

Note that there is NO  $G(x, \xi)$ .

The trick is to consider the IVP with homog. initial conditions  $u(0) = u'(0) = 0$ , for which there is a  $G_I(x, \xi)$ .

$$G_I(x, \xi) = \begin{cases} 0 & 0 \leq x < \xi \\ \frac{\psi(\xi)\phi(x) - \phi(\xi)\psi(x)}{J(\phi, \psi)} & 0 \leq \xi < x \end{cases}$$

Here  $G_I(0, \xi) = \frac{\partial G_I}{\partial x}(0, \xi) = 0$ .

Then  $G_I(x, \xi)$  is continuous at  $x = \xi$  and

$$-p(\xi) \left[ \frac{\partial G}{\partial x} \right]_{x=\xi} = 1 \quad \text{with } J = pW$$

So,

$$u_I(x) = \int_0^1 G_I(x, \xi) f(\xi) d\xi = \frac{\phi(x)}{J(\phi, \psi)} \int_0^x \psi(\xi) f(\xi) d\xi - \frac{\psi(x)}{J(\phi, \psi)} \int_0^x \phi(\xi) f(\xi) d\xi$$

This solution  $u_I(x)$  satisfies  $Lu_I = f$  and

$u_I(0) = u_I'(0) = 0$ . To satisfy BC

$B_1 u = C_1$  at  $x=1$ ,  $B_2 u = C_2$  at  $x=l$ , we

put

$$u_{II}(x) = u_I(x) + \frac{C_1 \phi(x)}{B_1 \phi}$$

(Note  $u_{II}(0) = u_I'(0) = 0$ ,  $B_1 u_{II} = 0$ )

BC at  $x=0$ :

$$B_1 u_{II} = \underbrace{B_1 u_I}_{=0} + \frac{C_1 \cancel{B_1 \phi}}{\cancel{B_1 \phi}_{\neq 0}} = C_1$$

$\therefore B_1 u_{II} = C_1$  (BC at left  
at  $x=0$  is  
satisfied)