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$$u_{\underline{II}}(x) = u_{\underline{I}}(x) + \frac{c_1 \phi(x)}{B_1 \phi}$$

We know that $L\phi=0$, $Lu_{\underline{I}}=f \Rightarrow$

$Lu_{\underline{II}}=f$, i.e. $u_{\underline{II}}$ satisfies $Lu=f$. (Note that $B_1 \phi \neq 0$ since we chose ϕ to be independent of ψ).

Now we just need to verify that BC at $x=1$ is satisfied — when (+) holds.

Then $u_{\underline{II}}$ is a solution of BVP.

$$u_{\underline{II}}(1) = u_{\underline{I}}(1) + \frac{c_1 \phi(1)}{B_1 \phi} =$$

$$= \frac{\phi(1)}{J(\phi, \psi)} \underbrace{\int_0^1 \psi(\xi) f(\xi) d\xi}_{<\psi, f>} - \frac{\psi(1)}{J(\phi, \psi)} \int_0^1 \phi(\xi) f(\xi) d\xi$$

$$+ \frac{c_1 \phi(1)}{B_1 \phi} \quad <\psi, f> \stackrel{(+) \atop \text{from last lecture}}{=} [J(u, \psi)]_0^1$$

$$u_{\underline{\underline{I}}} (1) = \frac{\phi'(1)}{J(\phi, \psi)} [J(u, \psi)]_0' + \frac{c_1 \phi'(1)}{B, \phi} -$$

$$- \frac{\psi'(1)}{J(\phi, \psi)} \int_0^1 \phi f d\xi$$

$$u_{\underline{\underline{I}}}^1 (1) = \frac{\phi'(1)}{J(\phi, \psi)} [J(u, \psi)]_0' + \frac{c_1 \phi'(1)}{B, \phi} -$$

$$- \frac{\psi'(1)}{J(\phi, \psi)} \int_0^1 \phi f d\xi$$

Similarly,

$$B_2 u_{\underline{\underline{I}}} = B_2 \phi \left(\frac{[J(u, \psi)]_0'}{J(\phi, \psi)} + \frac{c_1}{B, \phi} \right) +$$

$$+ \frac{B_2 \phi^0}{J(\phi, \psi)} \int_0^1 \phi f d\xi = \begin{matrix} \text{can be} \\ \text{shown} \\ \text{using} \\ (** \text{ and } +) \end{matrix} C_2$$

Ex 3 $-u'' = f(x)$ $x \in (0,1)$ like in Ex 1

$$u'(0) = a, u'(1) = b$$

but $B^*u = c_1 \neq 0$

BVP is self-adjoint with $u_{01} = A$ arbitrary constant, as a non-zero solution of the homog. problem as well as v_i of the adjoint problem.

Solvability condition solution of BVP

$$\int_0^1 Af dx = [J(u, A)]' \quad \Rightarrow$$

\int_0^1 $|$
 u_i $|$
 v_i

$$v_i = A$$

$$J(u, v) = uv' - u'v$$

$$v = A = \text{const} \rightarrow J(u, v) = -u'v$$

$$\Rightarrow -A u'(1) + A u'(0) = A(a - b)$$

$"$ $"$
 b a

i.e.

$$\boxed{\int_0^1 f(x)dx = a - b}$$

solvability condition



Interpretation in terms of heat conduction:

for a steady (t independent) temperature profile to exist, total heat energy produced inside medium $\equiv \int_0^l f(x) dx = -\alpha - \beta =$ net heat flux leaving the ends.

Note We can write the given BVP as

$$-u'' = f(x) - \underbrace{\left(\int_0^l f(x) dx - (\alpha - \beta) \right)}_{\text{this is zero}}$$

with the same BCs

$$u'(0) = \alpha \quad u'(l) = \beta$$

Then the solvability condition of Fredholm Alternative (FA) is satisfied automatically.

Modified Green's Function

Consider the BVP

$$(1) \quad \begin{cases} Lu = f(x) \\ B_1 u = 0 \quad B_2 u = 0 \end{cases}$$

$$(2) \quad \begin{cases} Lu_H = 0 \\ B_1 u_H = 0 \quad B_2 u_H = 0 \end{cases}$$

Recall if (2) has only zero sol $\Rightarrow \exists G(x, \xi)$

Adjoint BVP:

$$(3) \quad \begin{cases} L^* u_H^* = 0 \\ B_1^* u_H^* = 0 \quad B_2^* u_H^* = 0 \end{cases}$$

Procedure for Solving BVPs

1) Check homog. problem (2)

if $u_H = 0$ is the only solution, then

BVP (1) can be solved in terms of $G(x, \xi)$

If a non-trivial sol^u to (2) exists,
then go step 2).

- 2) Find non-trivial solution to adjoint problem (3). Check solvability / consistency conditions. If it is not satisfied, the BVP (1) has no solution. If it is satisfied, then
- 3) Solve BVP for non-unique sol^us.
Use modified Green's functions.

Note Solvability condition:

u_H^* is the
same as v
before

$$\int_a^b [u_H^* L u - u L u_H^*] dx = 0$$

$\stackrel{0}{\nearrow}$
 $\stackrel{\parallel}{\sim}$
 $\stackrel{f}{\searrow}$

$$\int_a^b u_H^*(x) f(x) dx = 0$$

$$J(u, v) \Big|_a^b = 0$$

$\stackrel{\parallel}{\sim}$
 $\stackrel{u^*}{\sim}$

since BC
are homog.
for u

Q If solvability condition is satisfied,
does $G(x, \xi)$ exist?

\nearrow suppose it does.

$$\mathcal{L} G(x, \xi) = \delta'(x - \xi)$$

+ BCs

$$u = G$$

$$u = u_H^*$$

$$\int_a^b \left[u_H^* \underbrace{\mathcal{L} G}_{\text{if}} - G \mathcal{L}^* \overrightarrow{u_H^*} \right] dx = 0$$

since BCs are homog.

$$\delta(x - \xi)$$

$$\int_a^b u_H^*(x) \delta(x - \xi) dx = 0$$

$$u_H^*(\xi) = 0 \quad \forall \xi \in [a, b]$$

\downarrow with assumption that u_H^* is a non-trivial solution of the adjoint problem

$\Rightarrow G(x, \xi)$ does not exist.

modified Green's function

For now, let $L = L^*$, $B_i = B_i^{-1}$. Let $u_H(x)$ be a nontrivial solution of (2).

Normalize

$$\int_a^b u_H^2(x) dx = 1$$

Let $G_m(x, \xi)$ be modified Green's function:

$$\begin{cases} L G_m(x, \xi) = \delta(x - \xi) - u_H(x) u_H(\xi) \\ B_1 G_m = 0, \quad B_2 G_m = 0 \end{cases}$$

\ chosen to
repair where
system breaks
down