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Symmetry condition:

$$(3.32) \quad w(x) G_M^*(x, \xi) = w(\xi) G_M(\xi, x)$$

Note This is typical for symmetry of  $G$  with weight  $w$ .

Result: solution of BVP (3.28) is

$$(3.33) \quad u(x) = \sum_{j=1}^k \mu_j u_{0j}(x) + \int_0^1 G_M(x, \xi) f(\xi) d\xi -$$

$u_N$                                $u_f$

$$- \left[ \int_{\xi=0}^{\xi=1} (u(\xi), \frac{G_M(x, \xi)}{w(\xi)}) \right]_{\xi=0}^{\xi=1}$$

( derivative are  
wrt  $\xi$ )

$u_C$

Note:  $G_M(x, \xi)$  is unique  $\Rightarrow u_f$  and  $u_C$  are unique

$u_N$ :  $M_j$  are arbitrary constants, expressing the non-uniqueness of the solution to the BVP (3.28).  $u_N$  is a solution of the homogeneous problem.

We can look for a minimum norm solution using the least squares method. This can be done by requiring

$$(3.34) \quad \langle u, u_{0i} \rangle = 0 \quad i=1, \dots, k$$

This is possible since  $\langle G_M(x, \xi), u_{0i}(x) \rangle = 0$   
 $i=1, \dots, k$

$u_f$  satisfies

$$Lu_f = f(x) - \sum_{j=1}^k v_j(x) \langle v_j, f \rangle$$

(3.35)

$$B_i u_f = 0 \quad i=1, \dots, n$$

The solvability condition for this BVP is satisfied automatically (HW - check).

The result for  $u_f$  in (3.33) follows (formally) on applying  $L$  and  $\delta_i$  for  $y_f$ .

$u_c$  satisfies

$$Lu_c = \sum_{j=1}^k v_j(x) [J(u, v_j)]_0' \quad (3.36)$$

$$B_i u_c = c_i \quad i=1, \dots, n$$

The solvability condition is satisfied automatically, i.e.  $\langle v_i, \text{RHS of ODE} \rangle = \langle v_i, f \rangle = [J(u, v_i)]_0'$

Adding (3.35) and (3.36), since the solvability condition for (3.28) is  $\langle v_i, f \rangle = [J(u, v_i)]_0'$ , we see that  $u_f + u_c$  satisfies (3.28).

Indeed,

$$L(y_f + u_c) = Ly_f + Lu_c =$$

$$= f(x) - \sum_{j=1}^k v_j(x) \langle v_j, f \rangle + \\ + \sum_{j=1}^k v_j(x) [J(u, v_j)]_0' =$$

$$= f(x) - \sum_{j=1}^k v_j(x) \left\{ \underbrace{\langle v_j, f \rangle - [J(u, v_j)]_0'}_{\text{so since this is a solvability condition}} \right\} =$$

$$= f(x)$$

$$\therefore L(u_f + u_c) = f(x)$$

$$B_i (u_f + u_c) = B_i \underset{0}{u_f} + B_i \underset{c_i}{u_c} = 0 + c_i = c_i$$

$$\therefore B_i (u_f + u_c) = c_i$$

Ex  
u<sub>f</sub> is

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi \quad (+)$$

$$L = a_0 u'' + a_1 u' + a_0$$

for a special case of  $n=2$ , general  $a_i(x)$   
 $w(x) = 1$ , separated BCs. Then

$$f(x, \xi) = \begin{cases} \frac{u_1(\xi) u_2(x)}{a_0(\xi) W(\xi)} & a \leq \xi < x \leq b \\ \frac{u_1(x) u_2(\xi)}{a_0(\xi) W(\xi)} & a \leq x < \xi \leq b \end{cases}$$

Using (+) we can write

$$u(x) = u_2(x) \int_a^x \frac{u_1(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi + u_1(x) \int_x^b \frac{u_2(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi$$

(+) is  
the same

$$u'(x) = u_2' \int_a^x \frac{u_1(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi + u_1' \int_x^b \frac{u_2(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi +$$

$$+ \frac{u_2(x) u_1(x) f(x)}{a_0(x) W(x)} - \frac{u_1(x) u_2(x) f(x)}{a_0(x) W(x)}$$

$$u_2''(x) = u_2'' \int_a^x \frac{u_1(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi + u_1'' \int_x^b \frac{u_2(\xi) f(\xi)}{a_0(\xi) W(\xi)} d\xi +$$

$$+ \frac{(u_2'(x)u_1(x) - u_1'(x)u_2(x))}{a_0(x)W(x)} f(x)$$

$\underbrace{\phantom{...}}$

$$= \frac{f(x)}{a_0(x)}$$

Check  $B_i u = 0$  and  $L u = f$ . Note

$$Lu_i = 0, \quad B_1 u_1 = B_2 u_2 = 0.$$

$$B_1 u = B_1 u_1 \int_a^b \underbrace{\frac{u_2 f}{a_0 W}}_{\text{const}} d\xi =$$

$\stackrel{\text{BC at } x=a}{}$

$$\int_a^a \dots = 0$$

$$= B_1 u_1 \cdot \text{const} = 0$$

"  
0

Similarly for  $B_2$ :

$$B_2 u = B_2 u_2 \int_a^b \underbrace{\frac{u_1 f}{a_0 W}}_{\text{const}} d\xi =$$

$\stackrel{\text{BC at } x=b}{}$

$$B_2 u_2 = 0 \quad 0$$

$$Lu = a_0 u'' + a_1 u' + a_0 u = \underbrace{L u_2}_{=0} \int_a^x \frac{u_1 f}{a_0 W} d\xi +$$

$$+ \underbrace{L u_1}_{\stackrel{\Rightarrow}{=0}} \int_x^b \frac{u_2 f}{a_0 w} d\xi + a_0 \cdot \frac{f}{a_0} = f$$

In general, consider BVP

$$Lu = f \quad \text{ODE}$$

$$B_i u = c_i \quad \text{BC}$$

① Consider the homog. problem

$$Lu = 0$$

$$B_i u = 0$$

Does this problem have any non-zero solution?

NO

YES

Case (a) of  
Fredholm Altern. Thm

Case (b) of  
Fredholm Altern. Thm

Case (a) the homog. problem has only the zero solution. there is a Green's function, which we might be able to construct.  $f(x, \xi)$

$u = f(x, \xi)$  is a solution of

$$Lu = \delta(x - \xi)$$

$$B_i u = 0$$

$G(x, \xi)$  is unique.

The solution to the original BVP exists and unique.

used in practice

$$u = \int_a^b G(x, \xi) f(\xi) d\xi + \sum_{i=1}^n v_i \tilde{u}_i$$

(                                  )

$$u_f \qquad \qquad \qquad \qquad \text{or } [J_\xi(u, G)]_a^b$$

used in theory

where  $\tilde{u}_i$  are linearly independent solutions of homog. ODE  $L\tilde{u}_i = 0$ , BCs are w/  $c_i$ .

let us also consider the case when problem is self-adjoint :  $L = L^* \Rightarrow G(x, \xi) = G(\xi, x)$

$\{L, D_B\}$

if  $w(x) = 1$ .