

Introduction

Linear 2-point BVP for ODE

$$\begin{array}{c} + \\ a \quad b \end{array}$$

An n^{th} order linear differential operator is of the form

$$L = \sum_{j=0}^n a_j(x) \frac{d^{n-j}}{dx^{n-j}} =$$

$$= a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots +$$

$$+ a_{n-1}(x) \frac{d}{dx} + a_n(x)$$

L acts on function $u(x)$, usually $u(x) \in C^n(a, b)$: $u(x)$ and its first n derivatives are continuous on (a, b) .

The 2-point BVP consists of

- (i) differential equation $Lu = f(x)$, $x \in (a, b)$

L, f : given

$u(x)$: need to find

(ii) boundary conditions, n , of the form

$$\sum_{j=0}^{n-1} \alpha_{ij} \frac{d^j}{dx^j} u(a) + \beta_{ij} \frac{d^j}{dx^j} u(b) = c_i, \quad i=1, \dots, n$$

bounded operator

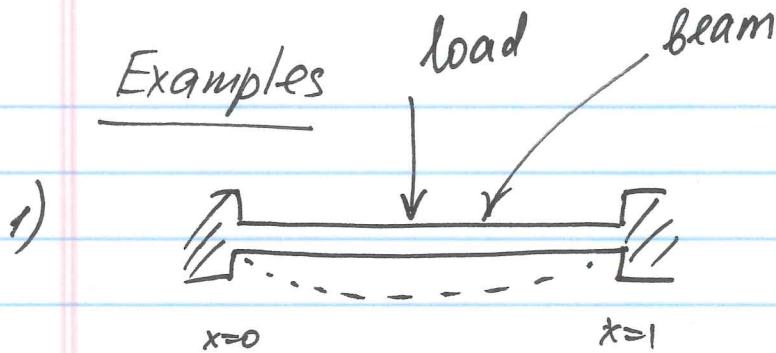
$\alpha_{ij}(x)$ are functions of x , but
 $\alpha_{ij}, \beta_{ij}, c_i$ are constants.

The problem is to find solution(s) for u ,
if they exist. If they do, to construct them.

If $f(x) \equiv 0$, the ODE is homogeneous,
i.e. there is no forcing

If all $c_i = 0$, $i=1, \dots, n$, the boundary conditions are homogeneous.

If ODE and BC are homogeneous,
we say that 2-point BVP is homogeneous.



$w(x)$: displacement
in vertical direction

Elastic problem:

$$\epsilon \frac{d^4 w}{dx^4} - \frac{d^2 w}{dx^2} = f(x) \quad x \in (0,1)$$

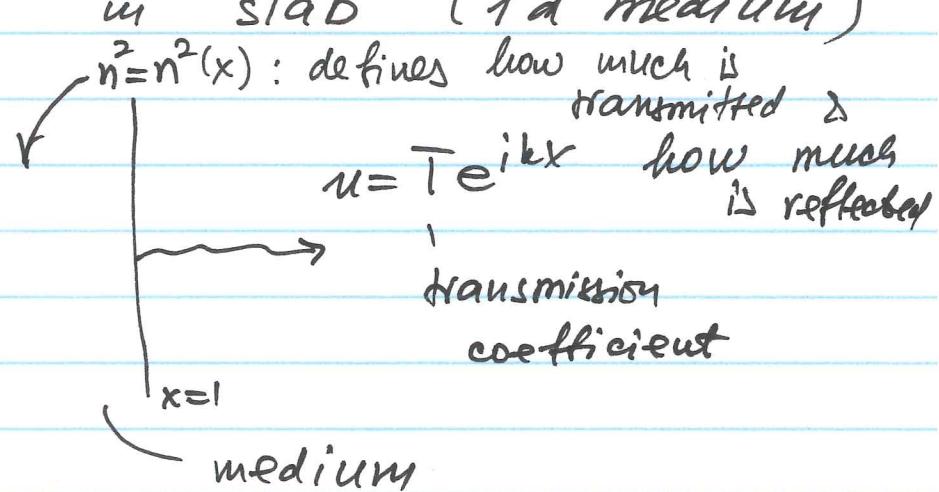
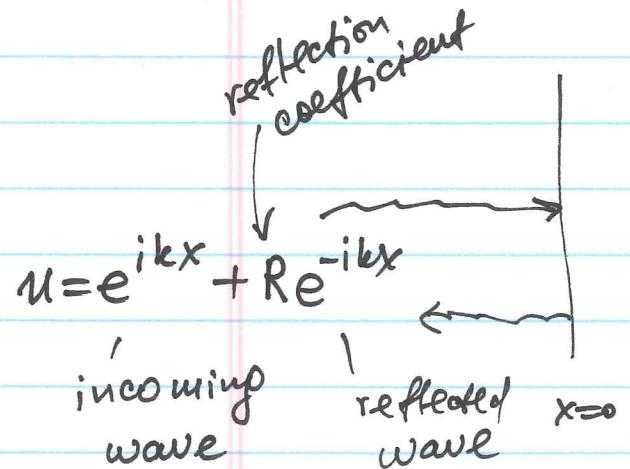
,
dimensionless
problem

$$\epsilon = \frac{\text{"stiffness"} }{\text{"tension"} } \ll 1$$

$$w(0) = w'(0) = 0, \quad w(1) = w'(1) = 0 : \quad \text{BCs for clamped beam}$$

f : load

2) linear wave in slab (1d medium)



$$\frac{d^2 u}{dx^2} + k^2 n^2(x) u = 0 \quad x \in (0, 1)$$

$$u'(0) + ik u(0) = 2ik$$

$$u'(1) - ik u(1) = 0$$

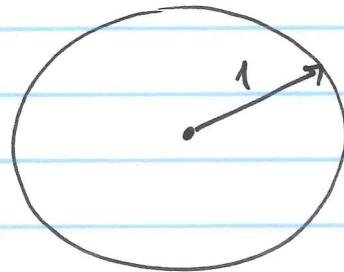
Electromagnetics and acoustics:

$$n^2 = \frac{c_0^2}{c^2(x)}$$

c: propagation speed of plane wave

3) Membrane / drumhead

Displacement $u(r)$, $r \in (0, 1)$



$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \gamma^2 u = 0$$

$$|u(0)| < \infty, \quad u(1) = 0$$

'membrane is clamped'

Spectrum λ_i :

'set of all eigenvalues λ_i '

We need to find all ϵ' -values λ_i for which there exist corresponding ϵ' -functions $u \neq 0$.

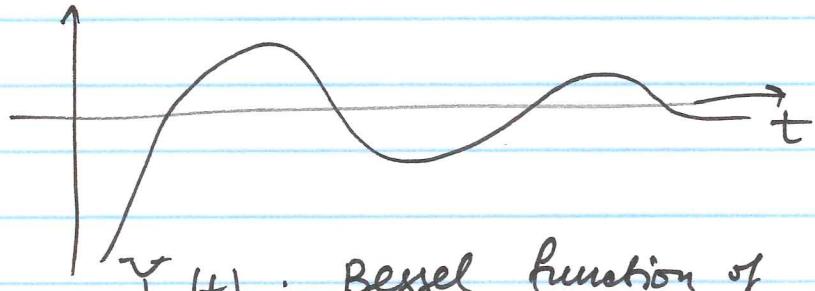
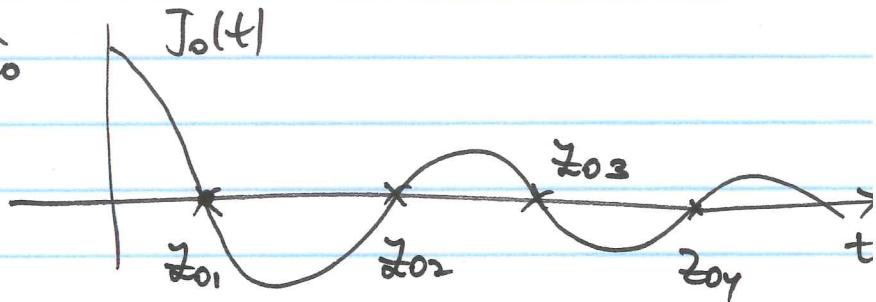
$$\lambda_i = J_0(z_{0i}; r) : \text{e'}\text{ functions}, \quad i=1, \dots$$

$J_0(z_{0i}) = 0$

$J_0(t)$: Bessel function of order 0 of 1st kind

z_{0i} : zeros of J_0

$$i=1, 2, \dots$$



$Y_0(t)$: Bessel function of 2nd kind

Note L and B_i are linear operators:

for any functions $u, v \in C^n(a, b)$,

✓ constants λ, μ :

$$L(\lambda u + \mu v) = \lambda L u + \mu L v$$

and

$$B_i(\lambda u + \mu v) = \lambda B_i u + \mu B_i v$$

$$Lu = f$$

If $f(x) = 0$, then ODE is homogeneous,
solutions of

$$Lu = 0$$

form a linear space: for $\forall u, v, \forall \lambda, \mu$

$$\begin{matrix} & 1 \\ Lu = 0, & Lv = 0 \end{matrix}$$

$$L(\lambda u + \mu v) = \lambda \underset{0}{\frac{d}{dx}} u + \mu \underset{0}{\frac{d}{dx}} v = 0$$

If $c_i = 0$ (i.e. $B_i u = c_i$), the B_i 's are homogeneous. Functions that satisfies $B_i u = 0$ also form a linear space.

For the problem to be "well-posed" the bounded operators B_i must be independent, i.e. any of the $B_i u$ must not be a linear combination of the other $B_j u$.

Equivalently, the $n \times m$ matrix $(\alpha_i | \beta_i)$ must be of full rank n .

Ex

$$x^2 u'' + x u' - u = x^2, \quad x \in (1, 2)$$

$$u(1) - 2u(2) = 0$$

$$u'(1) = 0$$

ODE is nonhomogeneous, BCs are homog.
 Problem is well-posed \Rightarrow there is a
 solution:

$$u(x) = \underbrace{\left(-\frac{7x}{9} - \frac{1}{9x} \right)}_{u_H(x)} + \underbrace{\frac{x^2}{3}}_{u_I(x)}$$

solution of
 homog. eq^z
 w/ RHS = 0

$$\mathcal{L}u_H = 0 \quad \mathcal{L}u_I = x^2$$

$$B_i(u_H + u_I) = 0 \quad i=1, 2$$

$$\begin{pmatrix} B_1 u \\ B_2 u \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{di}} \begin{pmatrix} u(1) \\ u'(1) \end{pmatrix} + \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{Bi,j}} \begin{pmatrix} u(2) \\ u'(2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$