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(Cont'd)

Ex $u'' - k^2 u = 0$

$B_1 u = u(0) - u'(0)$

$B_2 u = u(1) = 0$

$k=0$

$u'' = 0 \Rightarrow u(x) = Ax + B$ : general solution

$u'(x) = A$

$B_1 u = u(0) - u'(0) = B - A = 0$

$B_2 u = u(1) = A + B = 0$

$\Rightarrow A = B = 0$

$\therefore u(x) \equiv 0$ is the only solution of the homog. problem

$\therefore \exists\ G(x, \xi)$ when $k=0$.

Hence, $G(x, \xi)$ exists for $k \neq 0$.
Let's find it. BCs are separated, so we can use "$u_1, u_2$ approach". These solutions $u_1, u_2$ satisfy

$$u'' - k^2 u = 0$$

Let's consider the case when $k \neq 0$. The case $k=0$ is easier.

$e^{kx}$, $e^{-kx}$: linearly independent

$\sinh kx$, $\cosh kx$ : ——//——

$\sinh k(x-1)$, $\cosh k(x-1)$ : ——//——

$u_1$: $u_1 = A\sinh kx + B\cosh kx$

$B_1 u = u(0) - u'(0) = B - kA = 0 \Rightarrow B = kA$

$u_1 = A\sinh kx + kA\cosh kx = A(\sinh kx + k\cosh kx)$

Choose

$u_1 = \sinh kx + k\cosh kx$

$u_2$: $u_2 = A\sinh k(x-1) + \overset{0}{\cancel{B}}\cosh k(x-1)$

$B_2 u = u(1) = B = 0 \Rightarrow$ choose $u_2 = \sinh k(x-1)$

Put

$$G(x,\xi) = \begin{cases} A(\sinh kx + k\cosh kx)\sinh k(\xi-1) & 0 \le x < \xi \le 1 \\ A(\sinh k\xi + k\cosh k\xi)\sinh k(x-1), & 0 \le \xi < x \le 1 \end{cases}$$

Note : $G(x,\xi)$ is CTS at $x = \xi$ automatically.

at $x = \xi$ :

$$u'' - k^2 u = \delta(x-\xi)$$

Jump condition:

$$\int_{x=\xi^-}^{x=\xi^+} (u'' - k^2 u)\,dx = \int_{x=\xi^-}^{x=\xi^+} \delta(x-\xi)\,dx$$

$$\Rightarrow \left[\frac{dG}{dx}\right]_{x=\xi^-}^{x=\xi^+} = 1$$

$$\frac{dG}{dx} = \begin{cases} Ak(\cosh kx + k\sinh kx)\sinh k(\xi-1), & x<\xi \\ Ak(\sinh k\xi + k\cosh k\xi)\cosh k(x-1), & x>\xi \end{cases}$$

$$\left[\frac{dG}{dx}\right]_{x=\xi^-}^{x=\xi^+} = \left.\frac{dG}{dx}\right|_{x=\xi^+} - \left.\frac{dG}{dx}\right|_{x=\xi^-} =$$

$$= Ak(\sinh k\xi + k\cosh k\xi)\cosh k(\xi-1) -$$

$$- Ak(\cosh k\xi + k\sinh k\xi)\sinh k(\xi-1) = 1$$

Note

$$\sinh(a\pm b) = \sinh a\cdot\cosh b \pm \cosh a\cdot\sinh b$$

$$\cosh(a\pm b) = \cosh a\cdot\cosh b \pm \sinh a\cdot\sinh b$$

$$Ak\Big[(\sinh k\xi + k\cosh k\xi)(\cosh k\xi\cdot\cosh k - \sinh k\xi\cdot\sinh k) -$$

$$-(\cosh k\xi + k\sinh k\xi)(\sinh k\xi\cosh k - \cosh k\xi\cdot\sinh k)\Big] = 1$$

$$Ak\Big[k\cosh k\cdot\cosh^2 k\xi - \sinh k\cdot\sinh^2 k\xi - k\cosh k\sinh^2 k\xi +$$

$$+ \sinh k\cdot\cosh^2 k\xi\Big] = 1$$

$$Ak\Big[\cosh^2 k\xi\,(\sinh k + k\cosh k) - \sinh^2 k\xi\,(\sinh k + k\cosh k)\Big]$$

$$= 1$$

$$Ak\,\underbrace{(\cosh^2 k\xi - \sinh^2 k\xi)}_{=1}(\sinh k + k\cosh k) = 1$$

$$A = \frac{1}{k(\sinh k + k\cosh k)}$$

Hence,

$$G(x,\xi) = \begin{cases} \dfrac{(\sinh kx + k\cosh kx)\sinh k(\xi-1)}{k(\sinh k + k\cosh k)}, & 0 \le x < \xi \le 1 \\[2ex] \dfrac{(\sinh k\xi + k\cosh k\xi)\sinh k(x-1)}{k(\sinh k + k\cosh k)}, & 0 \le \xi < x \le 1 \end{cases}$$

Symmetry in $x$ and $\xi$ as expected since $\mathcal{L} = \mathcal{L}^*$.

We can consider asymptotic expansions of $G(x,\xi)$ for, say, $k$ being small.

$0 \le x, \xi \le 1$, $k$ is small $\Rightarrow$ $kx$, $k\xi$ are small & $k(x-1)$, $k(\xi-1)$

Note

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \quad k\cdot(1+\dots$$

$x < \xi$

$$G(x,\xi) = \frac{\left(kx + \frac{(kx)^3}{3!} + \dots\right)\left(k(\xi-1) + \dots\right)}{k\left(k + \dots + k\cdot 1 + \dots\right)} =$$

$$= \frac{k(x+1)\,k(\xi-1) + O(k^3)}{2k^2 + O(k^3)} \simeq \frac{(x+1)(\xi-1)}{2}$$

for small $k$

Similarly, for $x > \xi$.

**Ex** Problem #1, HW #4.

$$Lu = u'' - u = f(x)$$

$$B_1 u = u(1) - 2u(0) = c_1$$

$$B_2 u = u'(1) = c_2$$

$L = L^*$. BCs are mixed / unseparated, and $\mathcal{L} \neq \mathcal{L}^*$

Homogeneous problem:

$$u'' - u = 0$$

has linearly independent solutions $\cosh x$ and $\sinh x$.

The general solution is

$$u(x) = A\cosh x + B\sinh x$$

$$B_1 u = u(1) - 2u(0) = A\cosh 1 + B\sinh 1 - 2A = 0$$

$$B_2 u = u'(1) = A\sinh 1 + B\cosh 1 = 0$$

$$\begin{pmatrix} \cosh 1 - 2 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det(..) = \cosh 1(\cosh 1 - 2) - \sinh^2 1 = \cosh^2 1 - 2\cosh 1$$

$$- \sinh^2 1 = \underbrace{\cosh^2 1 - \sinh^2 1}_{=1} - 2\cosh 1 = 1 - 2\cosh 1 =$$

$$= 1 - (e^1 + e^{-1}) \neq 0$$

Hence, $A = B = 0$, i.e. homog. BVP has only trivial solution $\Rightarrow \exists\, G(x,\xi)$ and it is !.

Find $G(x,\xi)$. BCs are mixed.

Problem for $G(x,\xi)$:

$$LG(x,\xi) = \delta(x-\xi)$$

$$B;\ G = 0$$

for $x \neq \xi$, $G$ satisfies homog. ODE. The homog. solution will differ for $x < \xi$ and $x > \xi$.

Put

$$G(x,\xi) = \begin{cases} C\cosh x + D\sinh x & 0 \leq x < \xi \leq 1 \\ E\cosh(1-x) + F\sinh(1-x), & 0 \leq \xi < x \leq 1 \end{cases}$$

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