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Last time we obtained Green's function in the form

$$G(x, \xi) = \begin{cases} C \cosh x + D \sinh x, & 0 \leq x < \xi \leq 1 \\ \underset{\substack{E \\ \parallel \\ 2C}}{\cancel{E}} \cosh(1-x) + \overset{0}{\cancel{F}} \sinh(1-x), & 0 \leq \xi < x \leq 1 \end{cases}$$

BCs:

$$B_1 G = G(1) - 2G(0) = E - 2C = 0 \Rightarrow E = 2C$$

$$B_2 G = G'(1) = -F = 0$$

So,

$$G(x, \xi) = \begin{cases} C \cosh x + D \sinh x, & 0 \leq x < \xi \leq 1 \\ 2C \cosh(1-x), & 0 \leq \xi < x \leq 1 \end{cases}$$

Jump conditions:  $[G(x, \xi)]_{x=\xi} = 0$  : continuity at  $x = \xi$

$$\left[ \frac{dG}{dx}(x, \xi) \right]_{x=\xi} = 1$$

$$[G(x, \xi)]_{x=\xi^-}^{x=\xi^+} = 0$$

$$2C \cdot \cosh(1-\xi) - [C \cosh \xi + D \sinh \xi] = 0$$

$$\left[ \frac{dG(x, \xi)}{dx} \right]_{x=\xi^-}^{x=\xi^+} = 1$$

$$\frac{dG(x, \xi)}{dx} = \begin{cases} C \sinh x + D \cosh x, & 0 \leq x < \xi \leq 1 \\ -2C \sinh(1-x), & 0 \leq \xi < x \leq 1 \end{cases}$$

$$\Rightarrow -2C \sinh(1-\xi) - [C \sinh \xi + D \cosh \xi] = 1$$

$$\begin{pmatrix} 2 \cosh(1-\xi) - \cosh \xi & -\sinh \xi \\ -2 \sinh(1-\xi) - \sinh \xi & -\cosh \xi \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Use Cramer's Rule to solve this system to find C and D.

$$\Delta = \begin{vmatrix} 2 \cosh(1-\xi) - \cosh \xi & -\sinh \xi \\ -2 \sinh(1-\xi) - \sinh \xi & -\cosh \xi \end{vmatrix} =$$

$$= -\cosh \xi (2 \cosh(1-\xi) - \cosh \xi) - \sinh \xi (2 \sinh(1-\xi) + \sinh \xi) =$$

$$= -2 \cosh \xi \cosh(1-\xi) + \cosh^2 \xi - 2 \sinh \xi \sinh(1-\xi) - \sinh^2 \xi =$$

$$\cosh(a \pm b) = \cosh a \cdot \cosh b \pm \sinh a \cdot \sinh b$$

$$= \cosh^2 \xi - \sinh^2 \xi - 2 [\cosh \xi \cosh(1-\xi) + \sinh \xi \sinh(1-\xi)]$$

$$= 1 - 2 \cosh \left( \xi + (1-\xi) \right) = 1 - 2 \cosh 1$$

$$\Delta_1 = \begin{vmatrix} 0 & -\sinh \xi \\ 1 & -\cosh \xi \end{vmatrix} = \sinh \xi$$

$$\Delta_2 = \begin{vmatrix} 2 \cosh(1-\xi) - \cosh \xi & 0 \\ -2 \sinh(1-\xi) - \sinh \xi & 1 \end{vmatrix} = 2 \cosh(1-\xi) - \cosh \xi$$

Then

$$C = \frac{\Delta_1}{\Delta} = \frac{\sinh \xi}{1 - 2 \cosh 1}$$

$$D = \frac{\Delta_2}{\Delta} = \frac{2 \cosh(1-\xi) - \cosh \xi}{1 - 2 \cosh 1}$$

So,

$$G(x, \xi) = \begin{cases} \frac{\sinh(\xi - x) + 2 \cosh(1-\xi) \sinh x}{1 - 2 \cosh 1}, & 0 \leq x < \xi \leq 1 \\ \frac{2 \sinh \xi \cosh(1-x)}{1 - 2 \cosh 1}, & 0 \leq \xi < x \leq 1 \end{cases}$$

4

$G(x, \xi)$  is not symmetric, i.e.  $G(x, \xi) \neq G(\xi, x)$ .

Problem is not self-adjoint,  $L \neq L^*$ , that is why Green's function  $G(x, \xi)$  is not symmetric in  $x$  and  $\xi$ .

When we don't have a Green's function, i.e. homog. problem has a nonzero solution, we have resonant case.

Recall

modified Green's function

BVP  $Lu = f \quad x \in (0, 1)$

$$B_i u = C_i$$

$$\langle u, v \rangle = \int_0^1 uv \cdot w \, dx$$

Homog. problem has  $k \leq n$  non-zero lin. indep. solutions,  $u_{0i}, i=1, \dots, k$ :

resonant case.

Adjoint problem also has  $k$  <sup>non-zero</sup> solutions  $v_i, i=1, \dots, k$ .

In this case, there is no  $G(x, \xi)$ , but there is  $G_M(x, \xi)$  if solvability condition is satisfied.

We also normalize  $u_{0i}$  and  $v_i$ :

(3.29)

$$\langle u_{0i}, u_{0j} \rangle = \delta_{ij}$$

$$\langle v_i, v_j \rangle = \delta_{ij}$$

$G_M$ :

$$\left\{ \begin{array}{l} L G_M(x, \xi) = \delta(x - \xi) - w(\xi) \sum_{j=1}^k v_j(x) v_j(\xi) \\ B_i G_M = 0 \end{array} \right. \quad (3.30)$$

this term corresponds to solvability condition

Also, normalize:

$$\langle G_M(x, \xi), u_{0i} \rangle = 0 \quad i=1, \dots, k \quad (+)$$

$G_M^*$ :

$$\left\{ \begin{array}{l} L G_M^*(x, \xi) = \delta(x - \xi) - w(\xi) \sum_{j=1}^k u_{0j}(x) u_{0j}(\xi) \\ B_i^* G_M^* = 0 \end{array} \right. \quad (3.31)$$

also,

$$\langle G_M^*(x, \xi), v_i \rangle = 0 \quad i=1, \dots, k \quad (*)$$

Result 1:

$$w(x) G_M^*(x, \xi) = w(\xi) G_M(\xi, x) \quad (3.32)$$

Result 2

$$(3.33) \quad u(x) = \sum_{j=1}^k \mu_j u_{0j}(x) + \int_0^1 G_M(x, \xi) f(\xi) d\xi -$$

$\underbrace{\hspace{10em}}_{u_w} \qquad \qquad \qquad \underbrace{\hspace{10em}}_{u_f}$

$$- \left[ J_{\xi} \left( u(\xi), \frac{G_M(x, \xi)}{w(\xi)} \right) \right]_0^1$$

$\underbrace{\hspace{10em}}_{u_c}$

Let us show (3.32). Use Lagrange identity

$$\langle v, Lu \rangle = \langle L^* v, u \rangle + [J(v, u)]_0^1$$

with  $v(x) = G_M^*(x, \xi)$ ,  $u = G_M(x, \eta)$ .

$$\langle G_M^*(x, \xi), L G_M(x, \eta) \rangle = \langle L^* G_M^*(x, \xi), G_M(x, \eta) \rangle +$$

$$+ [J(G_M(x, \eta), G_M^*(x, \xi))]_0^1$$

$$= 0 \text{ since } B_1 G_M = 0, \quad B_1^* G_M^* = 0$$

Use (3.30) and (3.31) for  $L G_M$  and  $L^* G_M^*$ :

$$\int_0^1 G_M^*(x, \xi) \left( \delta(x-\eta) - w(\eta) \sum_{j=1}^k v_j(x) v_j(\eta) \right) w(x) dx$$

$$= \int_0^1 G_M(x, \eta) \left( \delta(x-\xi) - w(\xi) \sum_{j=0}^k u_{0j}(x) u_{0j}(\xi) \right) w(x) dx$$

So,

$$G_M^*(\eta, \xi) w(\eta) - w(\eta) \sum_{j=1}^k v_j(\eta) \underbrace{\int_0^1 G_M^*(x, \xi) v_j(x) w(x) dx}_{=0}$$

$\langle G_M^*, v_j \rangle = 0$   
from normalization  
condition (\*)

$$= G_M(\xi, \eta) w(\xi) - w(\xi) \sum_{j=0}^k u_{0j}(\xi) \underbrace{\int_0^1 G_M(x, \eta) u_{0j}(x) w(x) dx}_{=0}$$

$\langle G_M, u_{0j} \rangle = 0$   
from normalization  
condition (+)

Relabel  $\eta$  as  $x$

$$\Rightarrow w(x) G_M^*(x, \xi) = w(\xi) G_M(\xi, x) \quad (3.32)$$

Show (3.33). Lagrange identity w/  
 $u$  being a solution of BVP and

$$v(x) = \underbrace{G_M^*(x, \xi)}_{f(x)} \underbrace{\delta(x-\xi) - w(\xi)}_{/} \sum_{j=1}^k \underbrace{u_{oj}(x)}_{/} \underbrace{u_{oj}(\xi)}_{/}$$

$$\langle G_M^*(x, \xi), Lu \rangle = \langle L^* G_M^*(x, \xi), u \rangle +$$

$$+ \left[ J_x(u(x), G_M^*(x, \xi)) \right]_{x=0}^{x=1}$$

$$\int_0^1 G_M^*(x, \xi) \cdot f(x) w(x) dx = u(\xi) w(\xi) -$$

$$- w(\xi) \sum_{j=1}^k u_{oj}(\xi) \underbrace{\langle u_{oj}, u \rangle}_{= \text{const}} + \left[ J_x(u(x), G_M^*(x, \xi)) \right]_{x=0}^{x=1}$$

$\equiv M_j$

Use (3.32):

$$\int_0^1 G_M^*(x, \xi) f(x) w(x) dx = \int_0^1 w(\xi) G_M(\xi, x) f(x) dx$$

$$= w(\xi) \int_0^1 G_M(\xi, x) f(x) dx$$

Swap over  $x$  and  $\xi$ . Factor  $w(x)$  appears in each term. Then

$$u(x) = \sum_{j=1}^k u_{0j}(x) \cdot \mu_j + \int_0^1 G_M(x, \xi) f(\xi) d\xi - \left[ J_{\xi} \left( u(\xi), \frac{G_M(x, \xi)}{w(\xi)} \right) \right]_{\xi=0}^{\xi=1}$$

Indeed, swap over  $x$  and  $\xi$ :

$$\int_0^1 G_M^*(\xi, x) f(\xi) w(\xi) dx = u(x) w(x) -$$

$$- w(x) \sum_{j=1}^k u_{0j}(x) \underbrace{\langle u_{0j}, u \rangle}_{\mu_j} + \left[ J_{\xi} \left( u(\xi), G_M^*(\xi, x) \right) \right]_{\xi=0}^{\xi=1}$$

Use (3.32):

$$\int_0^1 G_M(x, \xi) f(\xi) w(x) d\xi = u(x) w(x) -$$

$$\frac{1}{w(x)} \left[ - w(x) \sum_{j=1}^k u_{0j}(x) \cdot \mu_j + \left[ J_{\xi} \left( u(\xi), \frac{w(x)}{w(\xi)} G_M(x, \xi) \right) \right]_{\xi=0}^{\xi=1} \right]$$

$$\Rightarrow u(x) = \int_0^1 G_M(x, \xi) f(\xi) d\xi + \sum_{j=1}^k u_{0j}(x) \cdot \mu_j -$$

$$-\left[ J_{\xi}(u(\xi), \frac{G_M(x, \xi)}{w(\xi)}) \right]_{\xi=0}^{\xi=1}$$

Jump condition for  $G_M(x, \xi)$

$$Lu = a_0(x)u^{(n)} + \dots + a_n(x)u = f$$

+ BCs

$$\text{Given } LG_M = a_0(x) \frac{d^4}{dx^4} G_M(x, \xi) + \dots + a_n(x) G_M(x, \xi) = \delta(x - \xi) - w(\xi) \sum_{j=1}^k v_j(x) v_j(\xi)$$

and homog. BCs

$$\int_{x=\xi^-}^{x=\xi^+} LG_M = \lim_{\epsilon \rightarrow 0} \int_{\xi-\epsilon}^{\xi+\epsilon} LG_M dx$$

function of  $x$ , so  $v_j(x) = \text{smooth function}$

$$\int_{\xi^+}^{\xi^+} v_j(x) dx = 0$$

$$a_0(\xi) \left[ \frac{d^{n-1}}{dx^{n-1}} G_M(x, \xi) \right]_{x=\xi^-}^{x=\xi^+} = 1$$

Ex 
$$Lu = -u'' = f(x), \quad x \in (0, 1)$$

$$B_1 u = u'(0) = C_1, \quad B_2 u = u'(1) = C_2$$
 weight  $w \equiv 1$

Recall  $J(u, v) = uv' - u'v$ . The problem is self-adjoint, with

$$u_0 = v_1 = 1 \quad k = 1$$

$G_M = G_M^*$  is symmetric

Solvability: 
$$\int_0^1 f(x) dx = C_1 - C_2$$

The problem for  $G_M(x, \xi)$  is

$$-G_M''(x, \xi) = \delta(x - \xi) - 1$$

$$G_M'(0, \xi) = G_M'(1, \xi) = 0$$

Normalization: 
$$\int_0^1 G_M(x, \xi) dx = 0$$

ODE 
$$G_M'' = 1 - \delta(x - \xi)$$

for  $x \neq \xi$ , the general solution of  $G_M'' = 1$  is of the form

$$G_M = \underbrace{\text{const} + \text{const} \cdot x}_{\text{general sol}^n \text{ of homog. eq}^n} + \frac{x^2}{2}$$

$G_M'' = 0$ 
 $G_M'' = 1$

particular solution of

So,

$$G_M(x, \xi) = \begin{cases} A + Bx + \frac{x^2}{2}, & 0 \leq x < \xi \leq 1 \\ C + Dx + \frac{x^2}{2}, & 0 \leq \xi < x \leq 1 \end{cases}$$

BCs:  $G_M'(0, \xi) = 0, G_M'(1, \xi) = 0$

$$\frac{dG}{dx}(x, \xi) = \begin{cases} B + x, & 0 \leq x < \xi \leq 1 \\ D + x, & 0 \leq \xi < x \leq 1 \end{cases}$$

$$G_M'(0, \xi) = 0 \Rightarrow B + 0 = 0 \Rightarrow \boxed{B = 0}$$

$$G_M'(1, \xi) = 0 \Rightarrow D + 1 = 0 \Rightarrow \boxed{D = -1}$$

$G_M(x, \xi)$  is continuous at  $x = \xi \Rightarrow$

$$A + \overset{0}{\cancel{B\xi}} + \frac{\xi^2}{2} = C + \underset{-1}{\cancel{D\xi}} + \frac{\xi^2}{2}$$

$$\boxed{A = C - \xi}$$

Jump condition at  $x = \xi$ :

$$-1 = \left[ \frac{dG}{dx} \right]_{x=\xi^-}^{x=\xi^+} = D + \frac{0}{\xi} - \left( \overset{0}{B} + \frac{0}{\xi} \right) \Rightarrow -1 = 1$$

this equation does not give any information

Note This always happens: the condition on

$\left[ \frac{d^{n-1}G}{dx^{n-1}} \right]$  gives no new information. It is

satisfied automatically.

Normalization condition is needed to determine  $A, B, C, D$  uniquely.

$$\int_0^1 G_M(x, \xi) dx = 0$$

$$\Rightarrow \int_0^{\xi} G_M(x, \xi) dx + \int_{\xi}^1 G_M(x, \xi) dx = 0$$

$0 \leq x < \xi$                        $\xi < x \leq 1$                       or                       $\xi < x \leq 1$

$$\int_0^{\xi} \left( A + \overset{0}{B}x + \frac{x^2}{2} \right) dx + \int_{\xi}^1 \left( C + \underset{-1}{D}x + \frac{x^2}{2} \right) dx = 0$$

$$A \frac{\xi^3}{6} + \frac{\xi^3}{6} + \left( Cx - \frac{x^2}{2} + \frac{x^3}{6} \right) \Big|_{x=\xi}^1 = 0$$

$$A \frac{x^3}{3} + \frac{x^3}{6} + C - C \frac{x}{\xi} - \frac{1}{2} + \frac{x^2}{2} + \frac{1}{6} - \frac{x^3}{6} = 0$$

$C = A + \xi$  : from continuity of  $G_M$  at  $x = \xi$

$$A \xi + (A + \xi)(1 - \xi) - \frac{1}{3} + \frac{\xi^2}{2} = 0$$

$$A \xi + A(1 - \xi) + \xi(1 - \xi) + \frac{\xi^2}{2} - \frac{1}{3} = 0$$

$$A = -\xi(1 - \xi) - \frac{\xi^2}{2} + \frac{1}{3}$$

$$A = -\xi + \frac{\xi^2}{2} + \frac{1}{3}$$

Then

$$C = A + \xi = -\xi + \frac{\xi^2}{2} + \frac{1}{3} + \xi = \frac{\xi^2}{2} + \frac{1}{3}$$

$$\Rightarrow C = \frac{\xi^2}{2} + \frac{1}{3} \quad B = 0 \quad D = -1$$

$$A = -\xi + \frac{\xi^2}{2} + \frac{1}{3}$$

Hence,

$$G_M(x, \xi) = \begin{cases} \frac{1}{3} - \xi + \frac{x^2 + \xi^2}{2}, & 0 \leq x < \xi \leq 1 \\ \frac{1}{3} - x + \frac{x^2 + \xi^2}{2}, & 0 \leq \xi < x < 1 \end{cases}$$

Note:  $G_M(x, \xi)$  is symmetric

Eq<sup>y</sup> (3.33) tells us that the solution for  $u$  is

$$u(x) = \underbrace{\text{const}}_{u_N} + \int_0^1 G_M(x, \xi) f(\xi) d\xi - \left[ J_{\xi} (u(\xi), G_M(x, \xi)) \right]_{\xi=0}^{\xi=1}$$

$u_c$ : Since  $J(u, v) = uv' - v'u''$   
 $= 0$ : BC on  $f_M$

$$u_c = - \left( u(1) \frac{\partial G_M}{\partial \xi}(x, 1) - u'(1) G_M(x, 1) \right) -$$

$$- \left( u(0) \frac{\partial G_M}{\partial \xi}(x, 0) + u'(0) G_M(x, 0) \right) \Bigg\} =$$

$= 0$  BCs on  $G_M$

$$= \underbrace{u'(1)}_{C_2} G_M(x, 1) - \underbrace{u'(0)}_{C_1} G_M(x, 0) = C_2 G_M(x, 1) -$$

$$- C_1 G_M(x, 0) = C_2 \left( \frac{x^2}{2} - \frac{1}{6} \right) - C_1 \left( \frac{1}{3} - x + \frac{x^2}{2} \right) =$$

$$= -\frac{C_1}{3} - \frac{C_2}{6} + C_1 x + (C_2 - C_1) \frac{x^2}{2}$$

$\underbrace{\quad}_{\text{const}}$ , can absorb with  $u_N$ .

We can check that  $L u_c = -u_c'' =$   
 $= [J(u, v)]_0^1 = C_1 - C_2$  and  $B_i u_c = C_i$   
 per (3.36).

$u_f$

For example,  $f(x) = -\sin \pi x$ . Some calculation of  $\int_0^1 G_M f dx$  gives

$$u_f = \int_0^1 G_M(x, \xi) f(\xi) d\xi = -\frac{1}{\pi^2} \sin \pi x -$$

$$-\frac{x^2}{\pi} + \frac{x}{\pi} + \underbrace{\frac{2}{\pi^3} - \frac{1}{6\pi}}_{\text{can put in } u_N}$$

We can check that

$$L u_f = -u_f'' = \sin \pi x + \frac{2}{\pi} = f(x) - \int_0^1 f(x) dx$$

and  $B_i u_f = 0$  per (3.35)

Since  $u_N$  is arbitrary, there are  $\infty$  many solutions. We can choose one having smallest norm. Then we will have the least squares / minimum norm problem.

Least squares solution:  $u = u_f + u_c$  ( $u_N = 0$ )  
 is

$$u = \underbrace{-\frac{\sin \pi x}{\pi^2} - \frac{x^2}{\pi} + \frac{x}{\pi}}_{u_f} + \underbrace{C_1 x + (C_2 - C_1) \frac{x^2}{2}}_{u_c} =$$

$$= -\frac{\sin \pi x}{\pi^2} - \frac{x^2}{2} \left( \frac{2}{\pi} - (C_1 - C_2) \right) + \left( C_1 + \frac{1}{\pi} \right) x$$

= 0 from solvability condition

So,

$$u = -\frac{\sin \pi x}{\pi^2} + \left( C_1 + \frac{1}{\pi} \right) x$$

$$u' = -\frac{\cos \pi x}{\pi} + C_1 + \frac{1}{\pi}$$

$$u'(0) = -\frac{1}{\pi} + C_1 + \frac{1}{\pi} = C_1$$

$$u'(1) = \frac{1}{\pi} + C_1 + \frac{1}{\pi} = \frac{2}{\pi} + C_1 = C_2: \text{ from solvability condition!}$$

Solvability condition:

$$\int_0^1 f(x) dx = C_1 - C_2$$

$$\int_0^1 (-\sin \pi x) dx = \frac{1}{\pi} \cos \pi x \Big|_0^1 = -\frac{1}{\pi} - \frac{1}{\pi} = -\frac{2}{\pi}$$

$$\Downarrow$$

$$\boxed{C_1 - C_2 = -\frac{2}{\pi}}$$

$$\text{or } \boxed{C_2 = C_1 + \frac{2}{\pi}}$$