

## 4 Eigen-function expansion methods for ODEs

Linear algebra analogy:

$$Ax = b \quad b, x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

Recall, a nonzero vector  $x \in \mathbb{R}^n$ ,  $x \neq 0$  is an eigenvector of matrix  $A$  with corresponding eigenvalue  $\lambda$  if  $Ax = \lambda x$ .

Given  $A$ , we find e'values  $\lambda_i$  by solving characteristic equation  $\det(A - \lambda I) = 0$ . Then for each  $\lambda_i$ , we find corresponding e'vector  $x_i$  by solving  $(A - \lambda_i I)x_i = 0$ .

Now,

- 1) If  $A$  has e'values  $\lambda_1, \lambda_2, \dots, \lambda_n$  with e'vectors  $x_1, x_2, \dots, x_n$  that span  $\mathbb{R}^n$ , then

$$A = S \Lambda S^{-1}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \dots \\ 0 & & \lambda_n \end{pmatrix} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

diagonalization of  $A$

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and

$$(4.1) \quad S = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{pmatrix} : \underline{\text{similarity}} \\ \underline{\text{transformation}}$$

If  $\exists k: x_k = 0 \Rightarrow \det \Lambda = 0 \Rightarrow \Lambda$  is singular  
 $\Rightarrow A$  is singular. In this case we can't  
 solve the problem uniquely.

If  $x_i \neq 0, i=1, \dots, n \Rightarrow \det \Lambda = \prod_{i=1}^n x_i \neq 0$

and so  $\det A \neq 0$ .

so  $A^{-1}$  exists (so  $Ax=b$  has a unique  
 solution) and  $(S^{-1})^{-1} = S$

$$A^{-1} = S \Lambda^{-1} S^{-1} \text{ where}$$

$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{x_1} & & 0 \\ & \frac{1}{x_2} & \dots \\ 0 & & \frac{1}{x_n} \end{pmatrix}$$

$$A = S \Lambda S^{-1}$$

$$\begin{aligned} A^{-1} &= (S^{-1})^{-1} \Lambda^{-1} S^{-1} \\ &= S \Lambda^{-1} S^{-1} \end{aligned}$$

The matrices  $A, A'$  (linear maps or linear operators) have a unique representation in terms of  $(x_i, \lambda_i)$ , the eigenvectors and eigenvalues of  $A$ .

The representation of  $A$  in terms of its eigensystem (eigenvalues and eigenvectors) is more complicated when one or more eigenvalues  $\lambda_i$  are repeated, but one can introduce generalized eigenvectors and Jordan normal form.

Note When eigenvalues are repeated, it is still possible to use (4.1) if for each eigenvalue  $\lambda_i$  with multiplicity  $m_i$  there are  $m_i$  lin. independent eigenvectors  $x_i$ .

2) For self-adjoint  $A$ ,  $A = A^T$  (real symmetric matrix), the representation of  $A$  in terms of its eigensystem is simpler. We also say about "spectral representation" of  $A$  in terms of eigenvalues & eigenvectors

(i) All eigenvalues  $\lambda_i$  are real.

(ii) There are real eigenvectors  $x_i$  that form

$S$  is unitary if  $S = (S^*)^T$

an orthonormal basis, provided that  
 $\lambda_i$  distinct,  $x_i x_j^T = \delta_{ij}$

(iii)  $S$  is orthogonal i.e.  $SS^T = I$  or  
 $S^{-1} = S^T$ . So, (4.1) becomes

$$(4.2) \quad A = S \Lambda S^T = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \dots + \lambda_n x_n x_n^T$$

(iv) If  $\lambda_i \neq 0$ ,  $i = 1, \dots, n$ , then  $A^{-1}$  exists

and

$$A^{-1} = S \Lambda^{-1} S^T$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & \dots & \\ 0 & & \ddots & \lambda_n \end{pmatrix}$$

similarly,

$$A^2 = S \Lambda^2 S^T = \lambda_1^2 x_1 x_1^T + \lambda_2^2 x_2 x_2^T + \dots + \lambda_n^2 x_n x_n^T$$

$$A^0 = I = x_1 x_1^T + \dots + x_n x_n^T$$

and for any analytic function

$$f(A) = S f(\Lambda) S^T$$