

Ex (Cont'd)

$$-u'' = \lambda u \quad x \in (0, l)$$

$$\text{BCs: } u(0) = 0, \quad u(l) = 0$$

Find λ 's values and associated u 'functions.

Case 2) 0

last time we found

$$\lambda = \left(\frac{n\pi}{l}\right)^2, \quad n=1, 2, \dots \quad \text{e}'\text{values}$$

$$u_n(x) = B_n \sin \frac{n\pi x}{l}, \quad n=1, 2, \dots \quad \text{e}'\text{function}$$

B_n are arbitrary constants unless we normalize the eigenfunction $u_n(x)$ by either choosing $B_n = 1$ or another common choice to require

$$\|u_n\|^2 = \int_0^l u_n^2(x) dx = 1$$

$$B_n^2 \int_0^l \sin^2 \frac{n\pi x}{l} dx = 1 \Rightarrow B_n^2 = \frac{2}{l}$$

$\overbrace{\qquad\qquad}^{\frac{l}{2}}$

$$B_n = \sqrt{\frac{2}{l}}$$

Case 2 = 0 $-u'' = 0 \Rightarrow u(x) = Ax + B$

$$u(0) = 0 \Rightarrow A \cdot 0 + B = 0 \Rightarrow B = 0$$

$$u(l) = 0 \Rightarrow A \cdot l = 0 \Rightarrow A = 0$$

$\therefore u(x) \equiv 0 \Rightarrow$ no nontrivial solutions

Hence, $\lambda = 0$ is not an eigenvalue.

Case 2 < 0 $-u'' = \lambda u$ Let $s = -\lambda > 0$

$$u'' - su = 0$$

$r^2 - s = 0$: characteristic equation

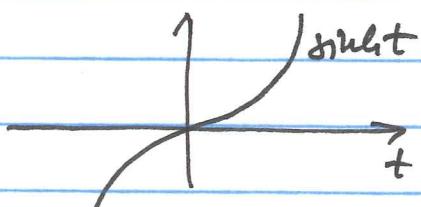
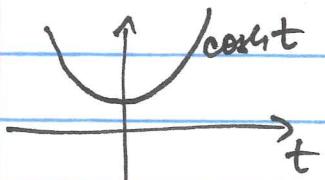
$$r^2 = s > 0 \Rightarrow r = \pm \sqrt{s}$$

$e^{\pm \sqrt{s}x}$ are lin. indep. solutions or

$\cosh \sqrt{s}x, \sinh \sqrt{s}x$ is another pair of lin. independent solutions

$$u(x) = A \cosh \sqrt{s}x + B \sinh \sqrt{s}x: \text{ general solution}$$

$$u(0) = 0 \Rightarrow 0 = A \cdot 1 + B \cdot 0 \Rightarrow A = 0$$



$$u(l) = B \underbrace{\sinh \sqrt{sl}}_{\neq 0} = 0 \Rightarrow B = 0$$

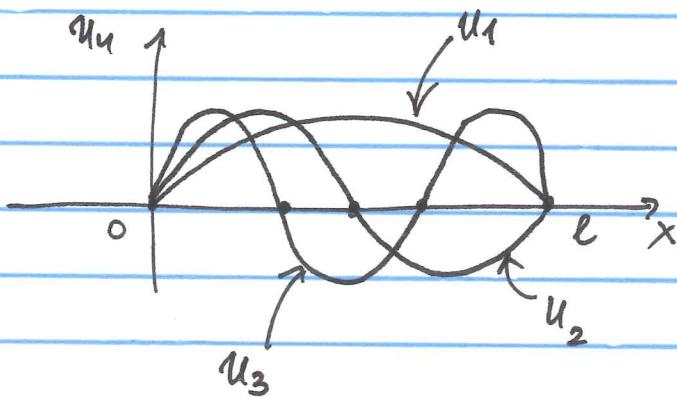
$\therefore u(x) \equiv 0 \Rightarrow l' \text{ values can't be } < 0.$

Therefore, the eigensystem is

$$(4.8) \quad (\lambda_n, u_n) = \left(\left(\frac{n\pi}{l} \right)^2, \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}, n=1, 3, \dots \right)$$

$$u_n = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}, \quad \lambda_n = \frac{n\pi}{l}$$

$$n=1 \quad \lambda_1 = \frac{\pi}{l} \quad u_1 = \sqrt{\frac{2}{l}} \sin \frac{\pi x}{l} \text{ has period } P=2l$$



$$n=2 \quad \lambda_2 = \frac{2\pi}{l} \quad u_2 = \sqrt{\frac{2}{l}} \sin \frac{2\pi x}{l} \quad P=l$$

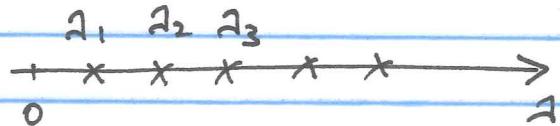
$$n=3 \quad \lambda_3 = \frac{3\pi}{l} \quad u_3 = \sqrt{\frac{2}{l}} \sin \frac{3\pi x}{l} \quad P=\frac{2}{3}l$$

Physically we have normal modes of a

stretched string, clamped at end $x=0, x=l$

In general, it can be shown that $u_n(x)$ has $n-1$ roots in $(0, l)$.

$$\lambda_n = \frac{n\pi}{l}, \quad n=1, 2, \dots$$



All eigenvalues are real and positive.

We can check that the BVP

$$-u'' = \lambda u \quad x \in (0, l)$$

$$u(0) = 0, \quad u(l) = 0$$

is self-adjoint. There is an adjoint sequence of eigenvalues and they are real. The eigenfunctions

$u_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}$ are orthonormal, i.e.

$$\begin{aligned} \langle u_i, u_j \rangle &= \int_0^l u_i(x) u_j(x) dx = \frac{2}{l} \int_0^l \sin \frac{i\pi x}{l} \sin \frac{j\pi x}{l} dx = \\ &= \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \end{aligned}$$

Functions $\{u_i(x)\}$ form a complete set, i.e.

any piecewise continuous function u has an expansion

$$\text{or } u = \sum_{i=1}^{\infty} d_i u_i \quad u \sim \sum_{i=1}^{\infty} d_i u_i : \text{Fourier series of } u$$

that converges to u in L^2 /mean square sense

2) Solution of a BVP for the same L as in 1)

We take

$$u'' = f(x) \quad (4.9)$$

$$u(0) = 0, \quad u(l) = 0$$

with homogeneous BCs.

Formally: look for an expansion of u in terms of eigenfunctions $u_n(x)$ of (4.8), i.e. find coefficients d_n s.t.

$$u(x) = \sum_{n=1}^{\infty} d_n u_n(x)$$

To do so, multiply both sides of ODE by u_n and then integrate $\int_0^l \dots dx$ and then

use properties of the eigenvalues.

$$\int_0^l u_n u'' = f u_n$$

$$\langle u_n, u'' \rangle = \int_0^l u_n u'' dx = \begin{matrix} \text{by parts} \\ \text{device} \end{matrix} [u_n u' - u_n' u]_0^l$$

$$+ \int_0^l u_n'' u dx \stackrel{=} {-u_n'' = \lambda u_n : \text{e'function}}$$

(here we are applying the Lagrange identity to

$$\langle u_n, L u \rangle = [J(u, u_n)]_0^l + \langle L^* u_n, u \rangle$$

Note that e'functions $u_n(x)$ satisfy homogeneous BCs:

$$u_n(0) = 0, \quad u_n(l) = 0$$

also

$$u(0) = 0, \quad u(l) = 0$$

$$\Rightarrow [u_n u' - u_n' u]_0^l = 0$$

Each terms in $[J(u, u_n)]_0^l$ vanishes since u_n and u satisfy homog. BCs.

$$\stackrel{=} {-\lambda_n} \int_0^l u_n u dx = -\lambda_n \langle u_n, u \rangle$$

On the RHS, we have $\langle f, u_n \rangle$, but f is given e'functions u_n are known \Rightarrow RHS is known.

Hence

$$-\lambda_n \langle u, u_n \rangle = \langle f, u_n \rangle$$

$$\Rightarrow \lambda_n = -\frac{\langle f, u_n \rangle}{\langle u, u_n \rangle} \Rightarrow \langle u, u_n \rangle = -\frac{\langle f, u_n \rangle}{\lambda_n}$$

$$\text{But if } u = \sum_{n=1}^{\infty} \alpha_n u_n \quad | \cdot u_n$$

then

$$\langle u, u_n \rangle = \int_0^l u_n \sum_{i=1}^{\infty} \alpha_i u_i dx \xrightarrow{\text{swap}} \int \alpha_n u_n dx$$

$$= \sum_{i=1}^{\infty} \alpha_i \underbrace{\int_0^l u_n u_i dx}_{\langle u_n, u_i \rangle} = \alpha_n$$

$$\langle u_n, u_i \rangle = \delta_{ni}$$

$$\Rightarrow \alpha_n = \langle u, u_n \rangle$$

Hence, formally (if some technical convergence issues are ok!)

$$u(x) = \sum_{n=1}^{\infty} d_n u_n = \sum_{n=1}^{\infty} \langle u, u_n \rangle u_n =$$

$$u(x) = \sum_{n=1}^{\infty} -\frac{\langle f, u_n \rangle}{\lambda_n} u_n(x)$$

(4.10)

is the representation/expansion of the solution of the BVP in terms of eigenvalues λ_n and eigenfunctions $u_n(x)$.