

Ex (Cont'd)

$$u'' = f(x)$$

$$u(0) = 0, \quad u(l) = 0$$

Last time we found an e' function representation of the solution (formally):

$$u(x) = \sum_{n=1}^{\infty} -\frac{\langle f, u_n \rangle}{\lambda_n} u_n(x)$$

where

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad n=1, 2, \dots$$

$$u_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}, \quad n=1, 2, \dots$$

} e' system

We also have a representation of the solution in terms of Green's function  $G(x, \xi)$ .

Check that

$$(4.11) \quad G(x, \xi) = \begin{cases} -\frac{x(\xi - l)}{l}, & 0 \leq x < \xi \leq l \\ -\frac{\xi(x - l)}{l}, & 0 \leq \xi < x \leq l \end{cases}$$

and so

$$u(x) = \int_0^l G(x, \xi) f(\xi) d\xi$$

Now (formally) equate the two representations of  $u(x)$ .

$$u(x) = \sum_{n=1}^{\infty} -\frac{\langle f, u_n \rangle}{\lambda_n} u_n(x) =$$

$$= \sum_{n=1}^{\infty} -\frac{u_n(x)}{\lambda_n} \int_0^l f(\xi) u_n(\xi) d\xi \stackrel{\text{swap}}{=} \sum_{n=1}^{\infty} \int_0^l$$

$$= \int_0^l \left( \sum_{n=1}^{\infty} \underbrace{\frac{u_n(x) u_n(\xi)}{-\lambda_n}} \right) f(\xi) d\xi =$$

$$= \int_0^l \underbrace{G(x, \xi)} f(\xi) d\xi$$

which suggests that

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(\xi)}{-\lambda_n}$$

(7.12)

Note

1) It turns out that (4.12) is valid: the RHS is the eigenfunction expansion of Green's function  $G(x, \xi)$ . But  $G(x, \xi)$  is known and constructed in (4.11); and as a piecewise continuous function has Fourier sine series — which we can construct. Since the e'functions  $u_n(x) = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi x}{\ell}$ , the Fourier series for  $G$  should be the expansion in terms of e'function (4.12) — IT IS.

$$G(x, \xi) = \sum_{n=1}^{\infty} g_n \sin \frac{n\pi x}{\ell}$$

Find  $g_n$ . Multiply both sides by  $\sin \frac{m\pi x}{\ell}$   
integrate  $\int_0^l$  (use orthogonality)

$$\int_0^l G(x, \xi) \sin \frac{n\pi x}{\ell} dx = \int_0^l \sum_{n=1}^{\infty} g_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx$$

$$\int_0^l \sum_{n=1}^{\infty} g_n \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx$$

$$= \sum_{n=1}^{\infty} g_n \underbrace{\int_0^l \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx}_{\frac{\ell}{2} \delta_{nm}}$$

$$\Rightarrow g_n = \frac{2}{l} \int_0^l G(x, \xi) \sin \frac{n\pi x}{l} dx$$

(use (4.11))

$$\Rightarrow g_n = -\frac{2l}{(n\pi)^2} \sin \frac{n\pi \xi}{l}$$

(check)

$$= \sqrt{\frac{l}{2}} \cdot \frac{u_n(\xi)}{-2n}$$

so, the Fourier sine series for  $G(x, \xi)$  is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(\xi)}{-2n}$$

same as in  
(4.12)

per the formal calculation /e'function expansion (4.12).

As an exercise: What changes if the BCs are inhomogeneous?

see  
HW# 6  
p. 2 (?)

$$B_1 u = u(0) = C_1 \quad \text{and} \quad B_2 u = u(l) = C_2$$

BVP:  $u'' = f(x), \quad x \in (0, l)$

$$B_1 u = u(0) = C_1, \quad B_2 u = u(l) = C_2$$

The  $\lambda$ 'values and  $u$ 'functions do NOT change!

$$(\lambda_n, u_n) = \left( \left( \frac{n\pi}{l} \right)^2, \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l} \right)$$

$$-u_n'' = \lambda_n u_n \quad u_n(0) = 0, \quad u_n(l) = 0$$

The change first appears:

$$\int_0^l u_n u'' dx = f$$

$$\int_0^l u_n u'' dx = \int_0^l u_n f dx$$

$$-\lambda_n \langle u_n, u \rangle$$

$$\text{LHS} = \langle u_n, u'' \rangle = [u_n u' - u_n' u]_0^l + \int_0^l u_n'' u dx$$

$$\xrightarrow{\neq 0}$$

$$-\lambda_n u_n$$

written in terms of  $c_i$

## 4.2 The Sturm-Liouville eigenvalue problem

The most general 2<sup>nd</sup> order linear differential operator that is formally self-adjoint with weight  $w$  is the

inner product  $\langle u, v \rangle = \int_0^1 uvw dx$ ; is the  
Sturm-Liouville operator

$$(4.13) \quad Lu = -\frac{1}{w} (pw')' + qu \quad x \in (0, 1)$$

$p, q, w$  are functions of  $x$ ,  $' = \frac{d}{dx}$ . For the "regular" Sturm-Liouville problem,  $p(x)$  and  $w(x)$  are non-zero throughout  $x \in [0, 1]$  and without loss of generality  $p(x)$  and  $w(x)$  are  $> 0$  for  $x \in [0, 1]$

( $p(x_0) = 0 \Rightarrow$  at least one of the linearly independent solutions of ODE is singular at  $x_0$ ).

for

(i) unmixed BCs

$$\beta_1 u = \alpha_1 u(0) + \beta_1 u'(0)$$

$$\beta_2 u = \alpha_2 u(1) + \beta_2 u'(1)$$

or

(ii) periodic BCs

$$B_1 u = u(0) - u(1)$$

$$B_2 u = u'(0) - u'(1)$$

with  $p(0) = p(1)$

the operator  $\mathcal{L} = \frac{1}{2} L, D_B^{-1}$  is self-adjoint.

The eigenvalue problem  $Lu = \lambda u$ ,  $B_i u = 0$  is Sturm-Liouville (S-L) for BCs (i) and sometimes but not always referred to as S-L for BCs (ii). Many of the same results hold for BCs (i) and (ii) but not all. In particular, for BCs (ii) there may be more than one eigenfunction for each eigenvalue).

Def A BVP is singular if  $p$  or  $w = 0$  at  $x \in [0, 1]$

OR

if the domain is not bounded

We will not consider the singular case.

Note: some "singular" problems do have the same features as "regular" problems.

The eigenvalue problems are very important in applications, e.g. "separation of variables" in PDEs.