

10/27/2017

$$Lu = -\frac{1}{w} (pu')' + qu$$

$$Lu = 2u$$

$$x \in (0, t)$$

$$B_i u = 0 \quad i=1, 2$$

$gw \rightarrow g \left\{ \begin{array}{l} \text{To follow convention, multiply the ODE} \\ \text{by } w \text{ and then replace } gw \text{ with } g \end{array} \right.$

$$(pu')' - qu - 2wu = 0$$

Take $p, w > 0$ continuous on $[0, t]$, g is continuous (p, w, g are real); B_i unmixed with all parameters real.

Theorem (λ 'values and e 'functions of the regular S-L problem).

- 1) There are infinitely many λ 'values $\lambda_1, \lambda_2, \dots$ and corresponding e 'functions u_1, u_2, \dots
- 2) The λ 'values are real and distinct and therefore can be ordered:

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots \quad \text{with } \lambda_n \rightarrow \infty \quad n \rightarrow \infty$$

i.e. there is the smallest λ 'value λ_1 and no largest λ 'value.

3) The e'functions can be chosen to form a complete orthonormal set wrt the inner product

$$\langle u, v \rangle = \int_0^1 uv w dx,$$

i.e.

$$\langle u_i, u_j \rangle = \int_0^1 u_i u_j w dx = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Proof

E.A. Coddington & N. Levinson

Theory of ordinary differential equations

1) The ∞ of e'values and e'functions is difficult to prove (at least lengthy).

$$Lu = -\frac{1}{w} (pu')' + qu \quad | \cdot w$$

Rename L to

$$Lu = -(pu')' + qu$$

For now, consider the possibility of z_n and u_n complex — we will show that $z_n \in \mathbb{R}$.

Generalize the inner product to

$$\langle u, v \rangle = \int_0^1 u \bar{v} w dx$$

- means complex conjugate.

2) Show that $\lambda_n \in \mathbb{R}$.

$$\text{for } (\lambda_n, u_n) \quad L u_n = \lambda_n w u_n$$

$$\Rightarrow (L - \lambda_n w) u_n = 0$$

$$\int_0^1 \bar{u} (L - \lambda w) u dx = 0$$

where $u = u_n$, $\lambda = \lambda_n$. Then

$$\int_0^1 \bar{u} ((pu')' - qu + \lambda w u) dx = 0$$

(
by parts once

$$u \bar{u} = |u|^2$$

$$\int_0^1 \bar{u} (pu')' dx \stackrel{\downarrow}{=} [\bar{u} (pu')]_0^1 - \int_0^1 p \underbrace{u' \bar{u}}_{|u''|} dx$$

Solving for λ , we get

$$\gamma = \frac{\int_0^1 p|u'|^2 + q|u|^2 - [pu' \bar{u}]_0'}{\int_0^1 |u|^2 w dx} \quad \text{is real}$$

1 / ^{real}
 real
 - / ^{real}
 = 0

\int_0^1
 $|u'|^2$
 $|u|^2$
 w dx
 > 0
 real
 ≠ 0

Note The boundary term $[pu' \bar{u}]_0' = 0$ if BCs are unmixed (check this). Also the boundary term $= 0$ if BCs are periodic with $p(0) = p(1) = 0$.

Now: $\phi, q, w \in \mathbb{R}$ gives $\gamma_n \in \mathbb{R}$.

Show that $\gamma_n \in \mathbb{R}$ using a different approach.

ALITER

(γ_n, u_n)

$$Lu_n = \gamma_n u_n w$$

(*)

$$B_i u_n = 0$$

Take complex conjugate of $Lu_n = \gamma_n u_n w$

$$L \text{ is real} \Rightarrow L = \bar{L}$$

$$w \text{ is real} \Rightarrow w = \bar{w}$$

The problem becomes

$$\left\{ \begin{array}{l} Lu_n = \gamma_n u_n w \\ \bar{L} \bar{u}_n = \bar{\gamma}_n \bar{u}_n \bar{w} \\ L \bar{u}_n = \bar{\gamma}_n \bar{u}_n \bar{w} \end{array} \right.$$

$$L \bar{u}_n = \bar{\lambda}_n \bar{u}_n w$$

λ_i : real

$$\lambda_i \bar{u}_n = 0$$

$$\lambda_i = \bar{\lambda}_i$$

since $p, q, w, \lambda_i \in \mathbb{R}$.

Form

$$u_n L \bar{u}_n - \bar{u}_n L u_n = (\bar{\lambda}_n - \lambda_n) w |u_n|^2$$

$$\bar{\lambda}_n \bar{u}_n w \quad \lambda_n u_n w$$

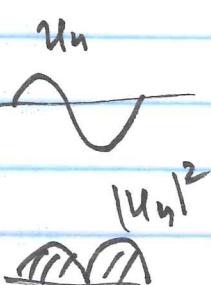
and integrate $\int_0^1 \dots dx$. We can show

that LHS = 0 by using integration by parts once and use BC (as we did above)

\Rightarrow $(\bar{\lambda}_n - \lambda_n) \int_0^1 |u_n|^2 w dx = 0$ but of the same

$$(\bar{\lambda}_n - \lambda_n) \underbrace{\int_0^1 |u_n|^2 w dx}_{>0} = 0$$

$$u_n \neq 0$$



$$\Rightarrow \bar{\lambda}_n - \lambda_n = 0 \text{ or } \bar{\lambda}_n = \lambda_n \Rightarrow \lambda_n \in \mathbb{R}$$

Note λ_n are distinct is difficult to prove
but we have shown that $\lambda_n \in \mathbb{R} \Rightarrow$

there is ordering

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

$\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ is also omitted (but not difficult to show).

We can show the following, easily:

for the regular S.-L. e'value problem, there is just one e'function that corresponds to each e'value (e'functions form a linearly independent set).

Proof

Suppose u_n and u_m are both e'functions w/ the same e'value λ_n . Show that u_n and u_m are linearly dependent, i.e.

$$W(u_n, \bar{u}_m) = 0$$

$$Lu_n = \lambda_n u_n w$$

$$Lu_m = \lambda_n u_m w$$

or

$$L\bar{u}_m = \lambda_n \bar{u}_m w$$

$$\Rightarrow \begin{aligned} (i) \quad & (L - \lambda_n w) u_n = 0 & (L - \underline{\lambda}_n w) \bar{u}_m = 0 \quad (ii) \\ & B_i u_n = 0 & B_i \bar{u}_m = 0 \end{aligned}$$

Multiply (i) by \bar{u}_m and subtract (ii) $\cdot u_n$

$$\Rightarrow \bar{u}_m (L - \lambda_n w) u_n - u_n (L - \lambda_n w) \bar{u}_m = 0$$

$$Lu = -(pu')' + qu, \text{ so}$$

$$(p\bar{u}_m)' u_n - (p u_n)' \bar{u}_m = 0 \quad \text{all other terms cancel}$$

is the perfect derivative

$$\Rightarrow \left(p(u_n \bar{u}'_m - \bar{u}_m u'_n) \right)' = 0 \quad \text{identity}$$