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Last time we obtained

$$\left(p (u_n \bar{u}_m' - \bar{u}_m u_n') \right)' = 0$$

\Rightarrow

$$(\dagger) \quad p W(u_n, \bar{u}_m) = \text{const}$$

$$\text{since } W(u_n, \bar{u}_m) = u_n \bar{u}_m' - u_n' \bar{u}_m$$

Evaluate $W(u_n, \bar{u}_m)$ at an end point $x=0$ or $x=1$. We have unmixed BCs
 $B_1: u=0 \Rightarrow \text{const} = 0$

$$p W = 0 \Rightarrow W(u_n, \bar{u}_m) = 0 \quad x \in [0, 1]$$

$\Rightarrow u_n$ and u_m are linearly dependent. \blacksquare

Note For other BCs including periodic BCs with the same L , we find that (\dagger)

$$p W(u_n, \bar{u}_m) = c \neq \text{const}$$

$$\Rightarrow \boxed{\bar{u}_m = (\text{arbitrary const}) u_n + \text{known function}}$$

In general, for this L ($S-L$, $n=2$), there are at least two linearly independent e 'functions for each e 'value.

For unmixed BCs, eigenfunction u_n are real without loss of generality.

(λ_n, u_n) : e 'value, e 'function pair
drop n

$$(pu')' - qu + \lambda wu = 0 \quad B_i u = 0$$

p, q, w : real, B_i : real

$\forall u = u_r + i u_i$, u_r, u_i : real

$$\Rightarrow (pu_r')' - q u_r + \lambda w u_r = 0 \quad B_i u_r = 0$$

$$(pu_i')' - q u_i + \lambda w u_i = 0 \quad B_i u_i = 0$$

$$(p\bar{u}')' - q\bar{u} + \lambda w\bar{u} = 0 \quad B_i \bar{u} = 0$$

Take
complex
conjugate

Hence, from argument after (+), the Wronskian of any two of u, \bar{u}, u_r and u_i ,

is zero \Rightarrow they are lin. dependent \Rightarrow
 $u \in \mathbb{R}$ wlog.

Note Sometimes it is convenient to let $u \in \mathbb{C}$

Recall

$n=2$

S-L operator

$$Lu \equiv -\frac{1}{w} (pu')' + qu$$

$p(x) > 0, w(x) > 0 \quad x \in [0, 1]$

p, q, w : real

Operator L is formally self-adjoint, i.e. $L=L^*$.

Regular

S-L

With unmixed BCs

BVP

$$B_1 u = \alpha_1 u'(0) + \beta_1 u(0)$$

$$B_2 u = \alpha_2 u'(1) + \beta_2 u(1)$$

the BVP is self-adjoint: $d = d^*$.

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There are ∞ many λ values λ_n that can be ordered:

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

λ functions u_n may be chosen to be orthonormal:

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

(*) For each λ value λ_n , there is just one linearly independent λ function u_n .

Most of these results hold for

$$Lu = -\frac{1}{w} (pu')' + qu$$

and periodic BCs:

$$B_1 u = u(1) - u(0)$$

$$B_2 u = u'(1) - u'(0)$$

The BVP is also self-adjoint: $L = L^*$

Above results are still valid but (*) is replaced with that there are at most two lin. independent φ -functions $u_n^{(1)}$ and $u_n^{(2)}$ corresponding to φ -value λ_n .

Corollary For the S-d. problem,

$$Lu = -\frac{1}{w} (pu')' + qu \quad | \cdot w$$

$$B_1 u = \alpha_1 u(0) - \beta_1 u'(0)$$

$$B_2 u = \alpha_2 u(1) - \beta_2 u'(1)$$

with $p(x) > 0$, $w(x) > 0$ for $x \in (0, 1]$,

if $q(x) > 0$ $x \in (0, 1]$

$$\alpha_1 \beta_1 \leq 0$$

$$\alpha_2 \beta_2 \geq 0$$

then

$$\lambda_n > 0$$

$$\neq \lambda$$

$$Lu = -(pu')' + qu$$

Proof $Lu = \lambda w u$ or $(L - \lambda w)u = 0$

$$Lu = - (pu')' + qu$$

$$\Rightarrow - (pu')' + qu = \lambda w u$$

$$\lambda = \lambda_n, \quad u = u_n$$

$$\int_0^1 u_n (L - \lambda_n \omega) u_n \, dx = 0$$

$$0 = \int_0^1 u_n \left(- (p u_n')' + q u_n - \lambda_n \omega u_n \right) dx =$$

by parts
once

$$= - [u_n p u_n']_0^1 + \int_0^1 \left(\underbrace{p u_n'^2}_{\geq 0} + \underbrace{q u_n^2}_{> 0} - \lambda_n \underbrace{\omega u_n^2}_{> 0} \right) dx$$

$$- [u_n p u_n']_0^1 = - p(1) u_n(1) u_n'(1) + p(0) u_n(0) u_n'(0) \quad (\equiv)$$

$$B_1 u = \alpha_1 u(0) + \beta_1 u'(0) = 0 \quad \xrightarrow{-\frac{\alpha_2}{\beta_2} u_n(1)} \quad u'(0) = -\frac{\alpha_1}{\beta_1} u(0)$$

$$B_2 u = \alpha_2 u(1) + \beta_2 u'(1) = 0 \quad \Rightarrow \quad u'(1) = -\frac{\alpha_2}{\beta_2} u(1)$$

$$\equiv \underbrace{p(1)}_{> 0} \cdot \underbrace{\frac{\alpha_2}{\beta_2}}_{\geq 0} u_n^2(1) - p(0) \underbrace{\frac{\alpha_1}{\beta_1}}_{\leq 0} u_n^2(0) \geq 0$$

From BCs and signs of d_s' and f_s' and solving for λ_n we get

$$\lambda_n = \frac{\int_0^1 p u_n'^2 + q u_n^2 dx + \text{boundary term}}{\int_0^1 u_n^2 w dx} > 0$$

$$\therefore \lambda_n > 0$$

4.3 Representation of a function in terms of orthonormal basis functions (completeness)

Let $\{u_n(x)\}$ be an orthonormal sequence of functions with weight $w(x)$, i.e.

$$\langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (4.17)$$

In this section, $\{u_n\}$ need not be e'functions of S-L. problem, but we say also applies to them.

The functions $u_n(x)$ are assumed to

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be smooth (typically C^∞) and square integrable (in $L^2(0,1)$, i.e.

$$\int_0^1 u_n^2 w dx < \infty$$