

The series representation of a function $f(x)$ is

$$u_j | \quad f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \quad (4.18)$$

where $\{a_n\}$ are generalized Fourier coefficients. Formally we can find a_j by forming $\langle u_j, f \rangle$

$$\langle u_j, f \rangle = \langle u_j, \sum_{n=1}^{\infty} a_n u_n \rangle = \sum_{n=1}^{\infty} a_n \langle u_j, u_n \rangle =$$

if we can
swap \int and \sum

(4.19)

by ortho-
normality of $\{u_n\}$

$$\sum_{n=1}^{\infty} a_n \delta_{jn} = a_j$$

so, $a_n = \langle u_n, f \rangle = \int_0^1 u_n f w dx$ certainly

exists if e.g. $f \in L^2(0,1)$, which we will assume.

Another way of determining coefficients $\{a_n\}$ is to minimize the error between f

and its representation - this gives the same result!

LEAST SQUARES DETERMINATION OF $\{a_n\}$

Replace (4.18) by a finite sum

$$f(x) \approx S_N = \sum_{n=1}^N a_n u_n(x)$$

Then determine $\{a_n\}$ such that the ' L^2 error' (least square error / mean-square deviation)

$$(4.20) \quad \varepsilon_N = \int_0^1 \left(f(x) - \sum_{n=1}^N a_n u_n(x) \right)^2 w(x) dx$$

is minimized.

\uparrow finite sum \rightarrow no problem in swapping order of \int and $\sum_{n=1}^N$

Now,

$$\begin{aligned} \varepsilon_N &= \int_0^1 f^2 w dx - 2 \sum_{n=1}^N a_n \underbrace{\int_0^1 f u_n w dx}_{= \langle f, u_n \rangle} + \\ &+ \sum_{j=1}^N \sum_{n=1}^N a_j a_n \underbrace{\int_0^1 u_j u_n w dx}_{= \langle u_j, u_n \rangle = \delta_{jn}} \end{aligned}$$

$$= \int_0^1 f^2 w dx - 2 \sum_{n=1}^N a_n \langle f, u_n \rangle + \sum_{j=1}^N a_j^2 =$$

↓

change $j \rightarrow n$

complete the square

$$(4.21) \quad = \sum_{n=1}^N \underbrace{(a_n - \langle f, u_n \rangle)^2}_{\geq 0} + \int_0^1 f^2 w dx - \sum_{n=1}^N \underbrace{\langle f, u_n \rangle^2}_{}$$

since we have
sum of squares ≥ 0

minimize it (hence
minimize E_N) by

choosing $a_n = \langle f, u_n \rangle$

These are FIXED for
given f and $\{u_n\} \Rightarrow$
we do nothing

so, the choice $a_n = \langle f, u_n \rangle, n=1, \dots, N$,
also minimizes E_N (the L^2 -error between f
and its series representation).

Note we get the same $a_n = \langle f, u_n \rangle$ if we change N (i.e. as N is increased, the "earlier" a_n 's stay the same).

BESSEL'S INEQUALITY

Def (4.20) of $\varepsilon_N \Rightarrow \varepsilon_N \geq 0$. Then the least square determination of $\{a_n\}$ (4.19) and (4.21)

\Rightarrow

$$\sum_{n=1}^N \langle f, u_n \rangle^2 \leq \int_0^1 f^2 w dx$$

(+) or $\sum_{n=1}^N a_n^2 \leq \int_0^1 f^2 w dx$ in terms of a_n

This holds for any N . The integral $\int_0^1 f^2 w dx$ converges since we assume $f \in L^2(0,1)$, i.e. f is square integrable. So,

$$P_N = \sum_{n=1}^N a_n^2 < \infty$$

for all N and P_N is monotone increasing as a sum of squares. Hence P_N converges as $N \rightarrow \infty$, and so (+) holds in the limit $N \rightarrow \infty$, i.e.

(4.22)

$$\sum_{n=1}^{\infty} a_n^2 \leq \int_0^1 f^2 w dx$$

BESSEL'S
INEQUALITY

Note This holds for ANY $\{u_n\}$ orthonormal set and any $f \in L^2(0,1)$ with $a_n = \langle f, u_n \rangle$.

We have a stronger result if $\{u_n\}$ and f are such that $E_N \rightarrow 0$ as $N \rightarrow \infty$, since we have

(i) strict equality in (4.22), which is known as Parseval's equality / Parseval's formula:

$$\sum_{n=1}^{\infty} a_n^2 = \int_0^1 f^2 w dx \quad \text{PARSEVAL'S EQUALITY}$$

(ii) (Terminology). We say that $\{u_n\}$ are "complete" — this is the def of completeness, that

$$\lim_{N \rightarrow \infty} E_N = 0$$

(iii) In (4.18) we have = in the mean square sense:

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \quad \text{with } a_n = \langle f, u_n \rangle \text{ in}$$

the sense that

$$0 = \int_0^1 \left(f(x) - \sum_{n=1}^{\infty} a_n u_n(x) \right)^2 w dx$$

Def The orthonormal functions $\{u_n\}$ are complete (wrt function f in some smoothness class, which we take to be $f \in C^0$, i.e. $f(x)$ is piecewise continuous, for $x \in (0,1)$) if with $a_n = \langle f, u_n \rangle$

$$\sum_{n=1}^{\infty} a_n^2 = \int_0^1 f^2 w dx \quad (4.23)$$

Given completeness and letting $N \rightarrow \infty$ in (4.21) implies that

$$\lim_{N \rightarrow \infty} \int_0^1 \left(f(x) - \sum_{n=1}^N a_n u_n(x) \right) w(x) dx = 0$$

so that $\sum_{n=1}^{\infty} a_n u_n(x)$ approximates or equals $f(x)$ in "mean square" sense.