

8/23/2017

## The formal adjoint (of $L$ )

The formal adjoint of  $L$  depends on  $L$  and on "inner product" which must be given.

Inner product maps two functions in linear function space of the problem to a scalar / constant.

e.g. (usual) if  $f, g \in C^n(a, b)$  the inner product of  $f$  and  $g$  is

$$\langle f, g \rangle = \int_a^b f g \, dx$$

To construct the formal adjoint, form the inner product  $\langle v, Lu \rangle$  for any functions  $u$  and  $v$  in the function space. Integrate by parts (apply Lagrange identity / Green's identity) until all the derivatives act on  $v$  (and none on  $u$ ), to find

$$\langle v, Lu \rangle \stackrel{\text{by parts}}{=} \langle L^*v, u \rangle + [J(u, v)]_a^b$$

then  $L^*$  is the formal adjoint.

If  $u$  and  $v$  satisfy homogeneous BC,  $B_i u = 0$ , then  $[J(u, v)]_a^b = 0$ , and the adjoint is ~~the~~ such that  
 formal

$$\langle v, Lu \rangle = \langle L^* v, u \rangle$$

Ex If  $L = a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x)$   
 (general 2<sup>nd</sup> order linear diff. operator)

and  $\langle f, g \rangle = \int_a^b f(x)g(x) dx$ ,

find  $L^*$ .

$$\langle v, Lu \rangle = \int_a^b v (a_0(x) u'' + a_1(x) u' + a_2(x) u) dx \quad \textcircled{=}$$

$$\int_a^b v a_1 u' dx \stackrel{\text{by parts}}{=} \left. \begin{array}{l} U = v a_1 \quad dV = u' dx \\ dU = (v a_1)' dx \quad V = u \end{array} \right|_a^b =$$

$$= [u \cdot v a_1]_a^b - \int_a^b u (v a_1)' dx$$

$$\int_a^b v a_0 u'' dx = \left. \begin{array}{l} U = v a_0 \quad dV = u'' dx \\ dU = (v a_0)' dx \quad V = u' \end{array} \right|_a^b =$$

$$= (u'va_0)|_a^b - \int_a^b (va_0)' u' dx \stackrel{\text{by}}{=} \text{parts again}$$

$$= \left| \begin{array}{l} \mathcal{U} = (va_0)' \\ d\mathcal{U} = (va_0)'' dx \end{array} \right| \begin{array}{l} dV = u' dx \\ V = u \end{array} =$$

$$= [u'va_0 - u(va_0)']_a^b + \int_a^b u(va_0)'' dx$$

$$\stackrel{\text{②}}{=} \int_a^b u [(va_0)'' - (va_1)' + a_2 v] dx +$$

$$+ [a_0 u'v - u(va_0)' + a_1 uv]_a^b$$

so,

$$\langle L^*v, u \rangle = \int_a^b \left( (va_0)'' - (va_1)' + a_2 v \right) u dx$$

so if  $u, v$  are  $\forall$  functions from  $C^2[a, b]$ , then the formal adjoint of  $L$  is

$$L^* = \frac{d^2}{dx^2} (a_0 \cdot) - \frac{d}{dx} (a_1 \cdot) + a_2$$

and

$$J(u, v) = a_0 u'v' - u(a_0 v)' + a_1 uv$$

Note that formal adjoint and adjoint are different notions.

$J(u, v)$  is called the "conjugate" of  $u, v$ . It is bilinear, i.e. it is linear in both  $u$  and  $v$ .

Def If differential operators  $L$  and  $L^*$ , its formal adjoint, are equal, i.e.  $L = L^*$ , then  $L$  is formally self-adjoint.

In above ex:

$$Lu = a_0 u'' + a_1 u' + a_2 u$$

$$L^*v = a_0 v'' + (2a_0' - a_1)v' + (a_0'' - a_1' + a_2)v$$

$$L = L^* \Leftrightarrow a_1 = 2a_0' - a_1 \quad (i)$$

$$a_2 = a_0'' - a_1' + a_2 \quad (ii)$$

$$(ii) \Rightarrow a_0'' - a_1' = 0 \Rightarrow a_0'' = a_1'$$

(i)  $\Rightarrow a_1 = a_0'$ , so (ii) is automatically satisfied

So, if  $a_1 = a_0'$ ,  $L = L^*$ . Then  $Lu$  can be simplified.

$$Lu = a_0 u'' + \underbrace{a_1 u'}_{a_0'} + a_2 u$$

hence,

$$Lu = \frac{d}{dx} \left( a_0 \frac{dy}{dx} \right) + a_2 u$$

or

$$L = \frac{d}{dx} \left( a_0 \frac{d}{dx} \right) + a_2$$

: formally self-adjoint operator

$n^{\text{th}}$  order case

$$L = \sum_{j=0}^n a_j(x) \frac{d^{n-j}}{dx^{n-j}}$$

$u, v \in C^n(a, b)$

' arbitrary from  $C^n$

Find formal adjoint operator  $L^*$ .

$$\langle v, Lu \rangle \stackrel{\text{def}}{=} \int_a^b v \cdot \sum_{j=0}^n a_j(x) \frac{d^{n-j}}{dx^{n-j}} u \, dx =$$

of inner product

$$= \sum_{j=0}^n \int_a^b v a_j(x) u^{(n-j)} \, dx$$

derivative of order  $n-j$

Integration by parts:

$$\int_a^b v a_j u^{(n-j)} \, dx = \left[ (a_j v) u^{(n-j-1)} - (a_j v)' u^{(n-j-2)} + \dots + (-1)^{n-j-1} (a_j v)^{(n-j-1)} u \right]_a^b +$$

$$+ (-1)^{n-j} \int_a^b u (a_j v)^{(n-j)} \, dx =$$

$$= \left[ \sum_{k=0}^{n-j-1} (-1)^k (a_j v)^{(k)} u^{(n-j-k-1)} \right]_a^b +$$

$$+ (-1)^{n-j} \int_a^b u (a_j v)^{(n-j)} \, dx$$

So,

$$\langle v, Lu \rangle = [J(v, b)]_a^b + \langle L^*v, u \rangle$$

where

$$J(v, u) = \sum_{j=0}^n \sum_{k=0}^{n-j-1} (-1)^k (a_j v)^{(k)} u^{(n-j-k-1)}$$

conjugate

$$L^*v = \sum_{j=0}^n (-1)^{n-j} \frac{d^{n-j}}{dx^{n-j}} (a_j v) =$$

$$= (-1)^n \frac{d^n}{dx^n} (a_0 v) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (a_1 v) + \dots$$

$$+ (-1) \frac{d}{dx} (a_{n-1} v) + a_n v \quad (\text{see pattern})$$

### Inner product

$J$  and  $L^*$  depend on inner product

$$\langle f, g \rangle = \int_a^b f g dx : \text{seen often}$$

In general, choice of inner product is motivated by the problem, and has

a "weight" function  $w(x)$ . So,

$$\langle f, g \rangle = \int_a^b f g w(x) dx$$

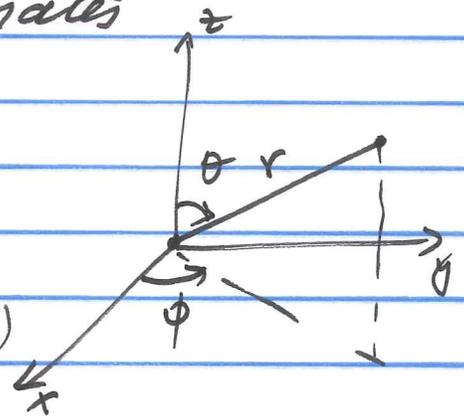
Often  $w$  is given by dimensionality of a physical context or coordinate system via volume element  $dV$ .

Ex 3D, spherical coordinates

$$dV = r^2 \sin \theta \, dr \, d\phi \, d\theta$$

For 2-point BVP in  $r$   
(after separation of variables)

weight is  $r^2$  ( $r^2 dr$ )



Ex 2D, polar coordinates

$$dV = "dA" = r \, dr \, d\theta$$

2-point BVP in terms of  $r$  (function of  $r$ )

weight is  $r$  ( $r dr$ )