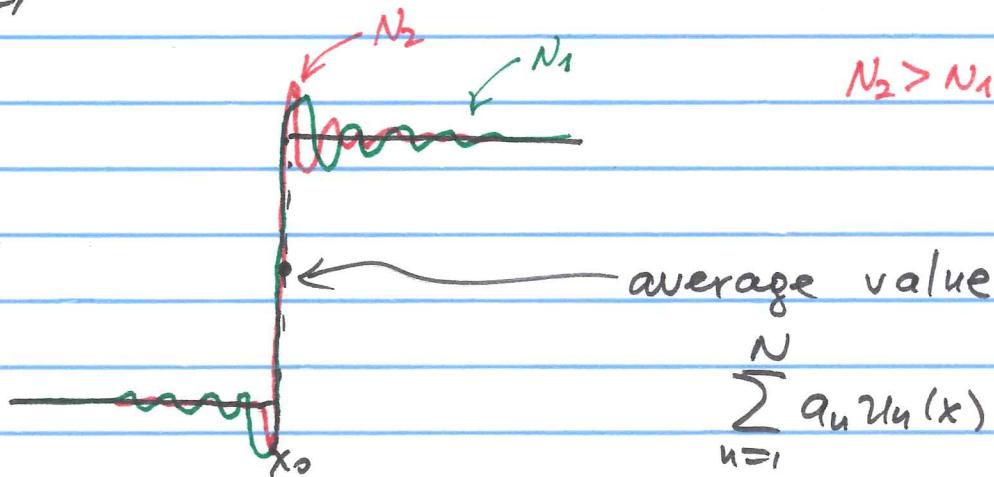
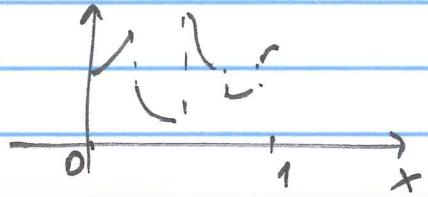


Q What smoothness conditions are needed for f ?

There are various results (all familiar in the context of Fourier series).

① $f \in L^2(0,1)$ (many functions) or
 (more restrictive) $f \in \text{pw } C^0(0,1)$ (piecewise continuous)
 (finite number of finite jump discontinuities on $(0,1)$ allowed)

Then we have mean square (L^2) convergence
 of $\sum_{n=1}^{\infty} a_n u_n(x)$ to $f(x)$



f has a finite jump discontinuity at x_0

as $N \uparrow$, the oscillations will become more localized, their amplitude \rightarrow away from x_0 but remaining the same around x_0 .

f has a "FINITE JUMP DISCONTINUITY"

u_n are smooth $\Rightarrow \sum_{n=1}^N a_n u_n(x)$ is smooth

$$\sum_{n=1}^N a_n u_n(x) \xrightarrow[N \rightarrow \infty]{} \frac{1}{2} (f(x_0^+) + f(x_0^-)) \text{ at } x = x_0$$

with 9% overshoot of $[f(x_0)]$ over decreasing jumps of f at x_0

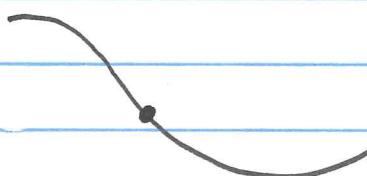
width as $N \rightarrow \infty$.

(2) $f \in C^0$ and f is pw C^1 , i.e. f is continuous on $(0, 1)$ and has continuous 1st derivative, except possibly at a finite # points $\in (0, 1)$ where derivative has finite jumps.



$\sum_{n=1}^N a_n u_n(x)$ converges pointwise to f

(3) $f \in C^1$ and pw. C^2 (f has continuous 1st derivative on $(0, 1)$ and (probably) finite jump(s) in second derivative at a finite # of points on $(0, 1)$)



$\Rightarrow \sum_{n=1}^{\infty} a_n u_n(x)$ converges absolutely and uniformly to f .

Now turn to basis functions $\{u_i\}$ that are eigenfunctions.

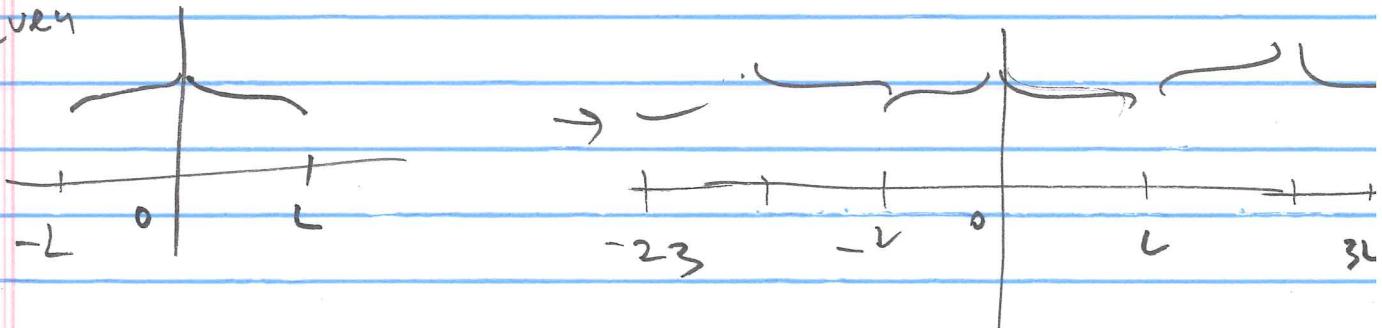
For expansions in terms of orthonormal eigenfunctions of

$$(L - \lambda_n w) u_n = 0$$

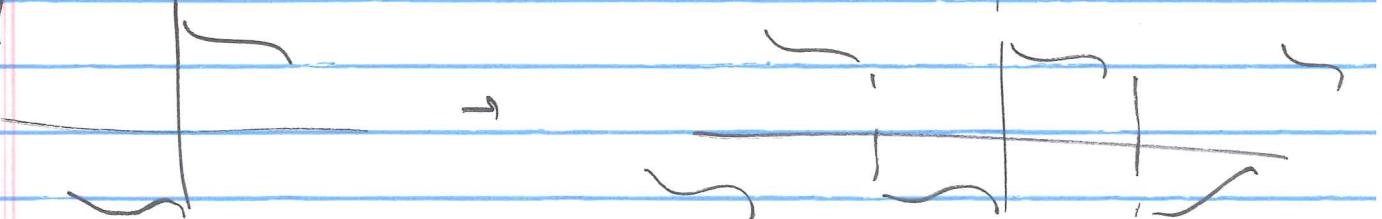
$$B_i u_n = 0$$

the function f that is being expanded must satisfy the same BCs as the eigenfunctions: $B_i f = 0$, for ② and ③ to hold at the end-points of the interval.

even



odd



Thm For expansion in eigenfunctions $u_n(x)$ such that $(L - \lambda_n w)u_n = 0$ $B_i u_n = 0$ if $f \in C^1$ and $f \in \text{pw } C^2$ with $B_i f = 0$, then

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

where $a_n = \langle u_n, f \rangle$ and the convergence of the sum to the function is absolute and uniform.

Note Uniform convergence enables us to swap $\int_0^1 \dots dx$ and $\sum_{n=1}^{\infty}$.

COROLLARY If $f, \{u_n\}$ satisfy the condition of the theorem, then Parseval's identity holds, i.e.

$$\sum_{n=1}^{\infty} a_n^2 = \int_0^1 f^2 w dx$$

D
As in (4.20)

$$e_N = \int_0^1 \left(f - \sum_{n=1}^N a_n u_n \right)^2 w dx$$

$\sum_{n=1}^{\infty} a_n u_n \rightarrow f$ uniformly as $N \rightarrow \infty$

to the \int_0^1 converges $\rightarrow 0$ uniformly.

so as $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$, which leads to Parseval's identity with $a_n = \langle u_n, f \rangle$.

COROLLARY If f, g and $\{u_n\}$ satisfy the conditions of the theorem, then

$$\langle f, g \rangle = \int_0^1 f g w dx = \sum_{n=1}^{\infty} a_n b_n$$

where

$$a_n = \langle u_n, f \rangle \quad b_n = \langle u_n, g \rangle$$

so $f = \sum_{n=1}^{\infty} a_n u_n$ and $g = \sum_{n=1}^{\infty} b_n u_n$

Proof

$$\langle f, g \rangle = \int_0^1 f g w dx = \int_0^1 f \left(\sum_{n=1}^{\infty} b_n u_n \right) w dx =$$

uniform convergence $\sum_{n=1}^{\infty} b_n \left(\int_0^1 f u_n w dx \right) = \sum_{n=1}^{\infty} a_n b_n$

$= a_n$

Q What is e'function expansion of $\delta(x-\xi)$ (in terms of orthonormal e'functions)?

We can try a formal expansion of $\delta(x-\xi)$ per (4.18) and (4.19), i.e.

$$\delta(x-\xi) = \sum_{n=1}^{\infty} a_n u_n(x)$$

with

$$a_n = \langle u_n, \delta(x-\xi) \rangle = \int_0^1 u_n(x) \delta(x-\xi) w(x) dx = \\ = u_n(\xi) w(\xi)$$

so, formally

$$\boxed{\delta(x-\xi) = w(\xi) \sum_{n=1}^{\infty} u_n(x) u_n(\xi)} \quad (*)$$

δ is symmetric

$$\delta(x-\xi) = \delta(\xi-x) \quad \text{or } w(x)$$

Q What (if any) is the convergence of $\sum a_n u_n(x)$ to $\delta(x-\xi)$? And in which sense?

There is not enough smoothness in $\delta(x-\xi)$ to guarantee convergence of the series in (6)