

4.4 Eigenfunction expansion for the regular Sturm-Liouville BVP (i.e. S-L diff operator with unmixed BCs).

Problem to solve is

$$(4.24) \quad \begin{cases} (L - \mu w)u = f(x) & x \in (0, 1) \\ B_i u = 0 \end{cases}$$

with

$$Lu = (-pu')' + qu, \quad B_i u = d_i u + \beta_i u' \text{ at } x=0, 1$$

$p, w > 0$, all functions are continuous, except possibly $f(x)$ which we take as piecewise continuous, i.e. pwC^0 .

μ is a parameter, and we take $\mu \neq \lambda_n, \forall n$, i.e. μ is away from all eigenvalues.

The eigen system (λ_n, u_n) is such that

$$(L - \lambda_n w)u_n = 0 \quad x \in (0, 1)$$

$$B_i u_n = 0 \quad i=1, 2$$

Now, f is pw $C^0 \Rightarrow u \in C^1$ and u is pw C^2
 \Rightarrow using Thm from last lecture, $u(x)$ has expansion

$$u(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

which converges absolutely and uniformly.

Recall $a_n = \langle u, u_n \rangle$

Multiply both sides of (4.04) by u_n and \int_0^1

$$u_n / (L - \mu w) u = f$$

$$\int_0^1 u_n L u dx - \mu \int_0^1 w u_n u dx = \int_0^1 f u_n dx$$

(1) (2) no w here

$$(1) \int_0^1 u_n L u dx = \int_0^1 u_n [- (p u')' + g u] dx =$$

alternative

by parts
twice

we can use

Lagrange

identity

w/

$$v = u u'$$

$$u = \text{sol } \frac{d}{dx}$$

BVP

$$= [p(u u'_n - u' u_n)]_0^1 + \int_0^1 u L u_n dx \quad (=)$$

$$= 0 \text{ if } B[u] = 0 \text{ use } B[u_n] = 0$$

$\therefore L = L^*$ formally self-adjoint

$\lambda u_n = \lambda_n w u_n \quad \therefore u_n$ is e' function
corresponding to e' value λ_n

$$(4.25) \quad \textcircled{1} \quad \lambda_n \int_0^1 u_n u_n w dx = \lambda_n \langle u_n, u_n \rangle = \lambda_n a_n$$

$$\textcircled{2} \quad = \mu \int_0^1 w u_n u_n dx = \mu \langle u_n, u_n \rangle = \mu a_n$$

so,

$$(\lambda_n - \mu) a_n = \int_0^1 u_n f dx$$

Since $\mu \neq \lambda_n$ for $\forall n$,

$$a_n = \frac{\int_0^1 u_n f dx}{\lambda_n - \mu} = - \frac{\int_0^1 u_n f dx}{\mu - \lambda_n}$$

and the series (eigenfunction) representation

is

$$(4.26) \quad u(x) = - \sum_{n=1}^{\infty} \frac{\int_0^1 u_n f dx}{\mu - \lambda_n} u_n(x)$$

Note

1) The weight w does not appear in $\int_0^1 u_n f dx$

2) The theorem in § 4.3 \Rightarrow the convergence of \sum in (4.26) is absolute and uniform since even with piecewise continuous $f(x)$, $pw.C^0(-\infty, \infty)$, $u \in C^1$ and $u \in pw.C^2(\sim)$

Notice that this also relies on the BCs being homogeneous, so that solution of BVP, u , satisfies the same BCs as e-function u_n . (Inhomogeneous BCs give slower convergence of \sum at endpoints $x=0, x=1$).

3) It is clear that the solution (4.26) diverges as $\mu \rightarrow \lambda_n$ for some n , i.e. $\mu \rightarrow$ an eigenvalue. (This implies that the given BVP (4.24) has non-zero solution for completely homogeneous problem ($f = g_i = 0$)). In this case there is no solution unless f is such that

SOLVABILITY
CONDITION

$$\int_0^1 u_j f dx = 0 \quad (*)$$

for that value $n=j$.

Then, when $\mu = \gamma_j$ and $\int_0^1 y_j f dx = 0$,

y_j is not determined and the general form of the solutions for u is

$$(4.27) \quad u(x) = \underbrace{C y_j(x)}_{\text{arb. const}} - \sum_{n=1}^{\infty} \frac{\int_0^1 u_n f dx}{\gamma_j - \gamma_n} u_n(x)$$

(*) is a necessary condition for a solution to the BVP to exist — it is also sufficient, and it is the solvability condition given by Fredholm alternative when $\mu = \text{eig val } \gamma_j$.

u_N

①: arbitrary multiple of a nontrivial solution of the homog. problem

② is response of u to the forcing f in the ODE

$$u_f \quad (u_c = 0)$$

This also agrees with the condition provided by the Fredholm alternative.

4) When $\mu \neq \lambda_n$ for any n , then BVP (4.24) has a Green's function $G(x, \xi; \mu)$ and the solution $u(x)$ ($\because B(u) = 0$) is

$$(+) \quad u(x) = \int_0^1 G(x, \xi; \mu) f(\xi) d\xi$$

Now we can justify swapping order of \sum and \int in (4.26) $\because \sum$ converges absolutely and uniformly.

$$u(x) = - \sum_{n=1}^{\infty} \frac{\int_0^1 u_n(\xi) f(\xi) d\xi}{\mu - \lambda_n} \quad u_n(x) = \begin{array}{l} \xi \text{ is a} \\ \text{dummy} \\ \text{variable} \end{array}$$

$$= \int_0^1 \left[- \sum_{n=1}^{\infty} \frac{u_n(\xi) u_n(x)}{\mu - \lambda_n} \right] f(\xi) d\xi \quad \begin{array}{l} \text{conv. is abs. \&} \\ \text{uniform} \Rightarrow \text{swap} \\ \sum \text{ and } \int \end{array}$$

Comparing with (+) $\Rightarrow (\because \text{solution is unique})$

(4.28)

$$G(x, \xi; \mu) = - \sum_{n=1}^{\infty} \frac{u_n(x) u_n(\xi)}{\mu - \lambda_n}$$

This is an eigenfunction expansion of $G(x, \xi; \mu)$.