

$$\frac{\dot{F}(t)}{F(t)} = \frac{u''(x)}{u(x)} = -2$$

Since $\frac{\dot{F}(t)}{F(t)}$ is a function of t alone and

$\frac{u''(x)}{u(x)}$ is a function of x alone, with two equal and x, t being independent variables, both sides have to CONSTANT, equal to a separation constant -2 .

so,

$$(4.32) \quad \dot{F}(t) + 2 F(t) = 0$$

and

$$(4.33) \quad u''(x) + 2 u(x) = 0$$

BCs:

$$\theta(0,t) = 0 \Rightarrow u(0) F(t) = 0 \rightarrow \boxed{u(0) = 0}$$

$$\theta(x,t) = u(x) F(t)$$

$$\theta(l,t) + C \theta_x(l,t) = 0 \Rightarrow$$

$$u(l) F(t) + C u'(l) F(t) = 0$$

$$[u(l) + C u'(l)] F(t) = 0 \Rightarrow \boxed{u(l) + C u'(l) = 0}$$

\times_0
in general

i.e.

$$(4.34) \quad \begin{aligned} B_1 u &= u(0) = 0 \\ B_2 u &= u(1) + c u'(1) = 0 \end{aligned}$$

(4.33) and (4.34) form a self-adjoint SL eigenvalue problem for $u(x)$ with $L = -\frac{d^2}{dx^2}$, B_i as given unmixed, and e' value λ .

so, from § 4.3, e'values λ_n are real, distinct, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and to each λ_n there is one orthonormal e'f ψ_n . These functions $\{\psi_n\}$ are complete and $\langle \psi_i, \psi_j \rangle = \delta_{ij}$.

Once (λ_n, ψ_n) $n=1, 2, \dots$ are known, (4.32) gives

$$\frac{dF}{dt} = -\lambda F(t) \Rightarrow F(t) \propto e^{-\lambda t}$$

$$\frac{dF}{F} = -\lambda dt, \quad F \neq 0 \quad \text{proportional}$$

$$\ln|F(t)| = -\lambda t + \tilde{C} \quad | \exp$$

$$F(t) = C e^{-\lambda t} \quad | F(t) = e^{-\lambda t} \cdot e^{\tilde{C}}$$

$$C = t$$

$$\text{or} \quad F_n(t) \propto e^{-\lambda n t}$$

$$\text{or} \quad F_n(t) = C_n e^{-\lambda n t}$$

Since the problem is linear and eigenfunctions are complete, the solution for θ is a linear superposition or series in terms of eigenfunctions

$$\theta(x, t) = \sum_{n=1}^{\infty} a_n u_n(x) e^{-\lambda_n t}$$

The coefficients $\{a_n\}$ are determined by the initial condition $\theta(x, 0) = f(x)$

at $t=0$

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \quad [\cdot u_j \int_0^1]$$

Orthogonality \Rightarrow

$$a_n = \langle f, u_n \rangle$$

Note since

$$\lambda_1 < \lambda_2 < \dots$$

the dominant behaviour of θ for large t ($t \rightarrow \infty$) is associated with the smallest eigenvalue λ_1 and

$$\theta(x, t) \sim a_1 u_1(x) e^{-\lambda_1 t} \quad \text{as } t \rightarrow \infty$$

provided

$$a_1 = \langle f, u_1 \rangle \neq 0$$

If $\lambda_1 > 0$, then $\theta \rightarrow 0$ as $t \rightarrow \infty$ (expected if $c > 0$) but if $\lambda_1 < 0$, then $\theta \rightarrow \infty$ as $t \rightarrow \infty$ which appears possible if $c < 0$.

Eigensystem:

$$u'' + \lambda u = 0 \quad x \in (0, 1)$$

$$u(0) = 0 \quad u(1) + cu'(1) = 0$$

c is given parameter and it is fixed.

ODE has two linearly independent solutions ($\lambda > 0$): $\sin \sqrt{\lambda}x$ and $\cos \sqrt{\lambda}x$

$$u(x) = A \sin \sqrt{\lambda}x + B \overset{1}{\underset{x}{\cos}} \sqrt{\lambda}x: \text{ general sol}$$

$$\text{BCs: } u(0) = 0 \Rightarrow A \cdot 0 + B \cdot 1 = 0 \Rightarrow B = 0$$

$$u(x) = A \sin \sqrt{\lambda}x \quad u' = A \sqrt{\lambda} \cos \sqrt{\lambda}x$$

$$u(1) + cu'(1) = 0$$

$$A \sin \sqrt{\lambda} + c A \sqrt{\lambda} \cos \sqrt{\lambda} = 0$$

$$A(\sin \sqrt{z} + C\sqrt{z} \cos \sqrt{z}) = 0$$

to

for a nontrivial
solution

$$\Rightarrow \sin \sqrt{z} + C\sqrt{z} \cos \sqrt{z} = 0$$

i. ℓ' 'values z_n are roots of

$$\sin \sqrt{z} + C\sqrt{z} \cos \sqrt{z} = 0 \quad | \quad \frac{1}{\cos \sqrt{z}}$$

or of

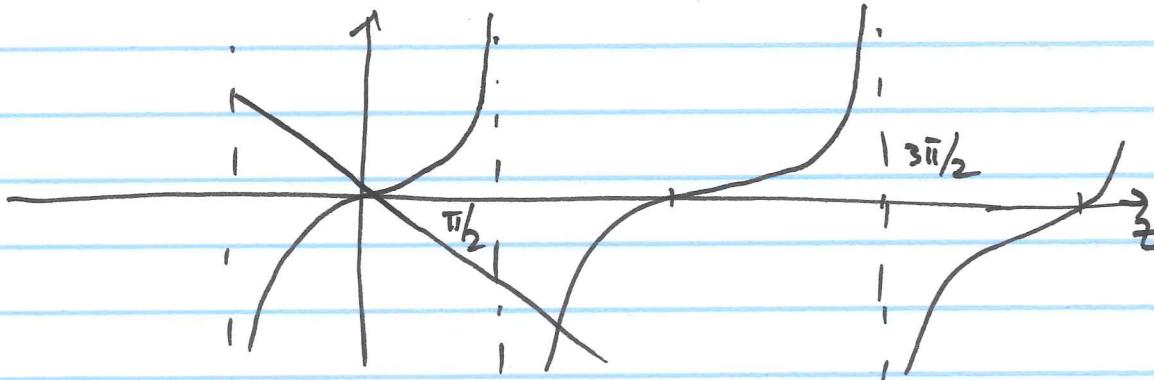
$$\tan \sqrt{z} = -C\sqrt{z}$$

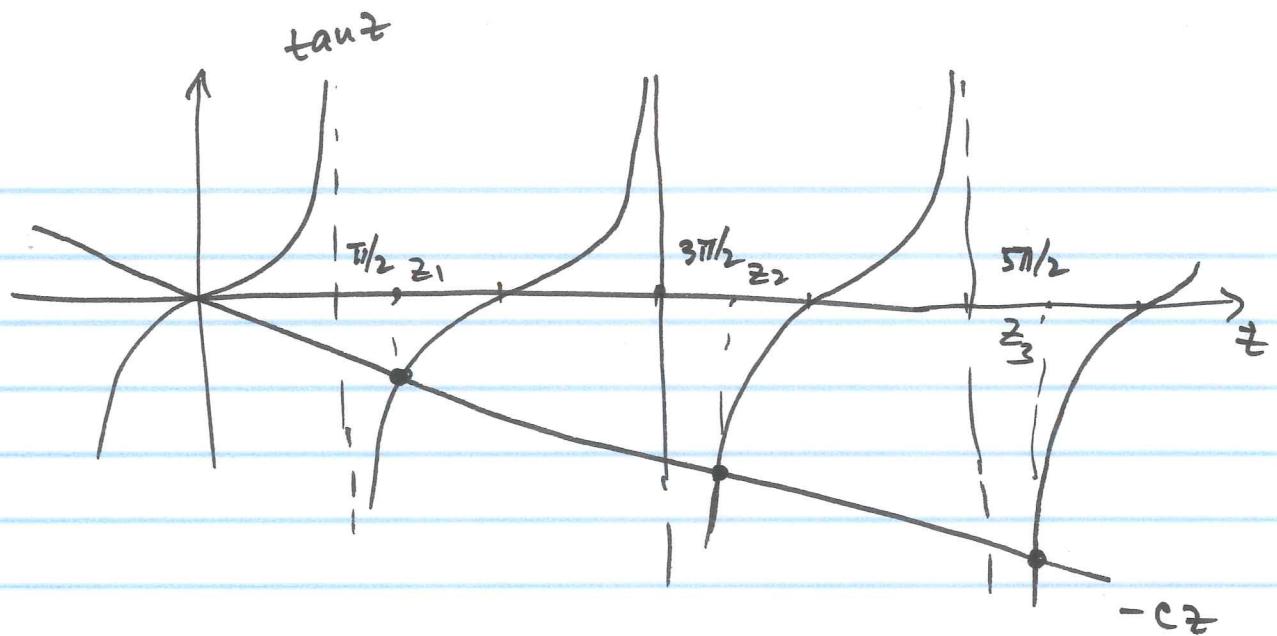
The roots are known to be real.

(i) $C > 0$

$z \geq 0$ Set $z = \sqrt{z}$ and look for $z \geq 0$.

$$\tan z = -Cz$$





Note $z_n > 0$ is a root $\Rightarrow -z_n$ is also a root

$z_n = z_n^2$, so we can just take $z_n > 0$.

Approximations for $n \gg 1$.

$$z_n \approx (2n-1) \frac{\pi}{2} + \frac{1}{c n^a} + \dots$$

$$\Rightarrow z_n \approx (2n-1)^2 \frac{\pi^2}{4} \quad \text{for large } n$$

$$\begin{aligned} &\stackrel{?}{=} (4.33) \Rightarrow u''=0 \Rightarrow u = Ax + B^0 \\ &\quad (u(0)=0 \Rightarrow 0=A \cdot 0 + B \Rightarrow B=0) \end{aligned}$$

$$u(1)+cu'(1)=0 \Rightarrow A \cdot 1 + c \cdot A = 0 \quad \text{or} \quad A(1+c)=0$$

if $A \neq 0$, it gives a function $u = Ax$ iff $c = -1$

Hence, $z=0$ is an e'value iff $c=-1$.

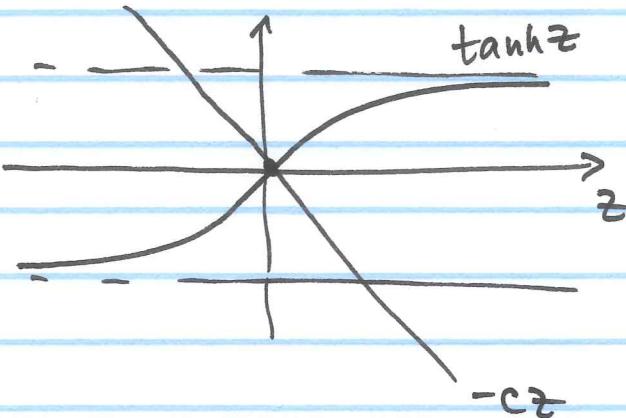
$$\underline{z \leq 0} \quad \sqrt{z} = iz, \quad z \text{ is real}$$

then

$$\tan(iy) =$$

$$= iy \operatorname{tanh} y \quad \text{or}$$

$$\operatorname{tanh} z = -cz$$



no roots here since the only root as we can see from the sketch is when $z=0$ and we consider this case ($z=0$) separately.

So, for $c > 0$, all e'values are positive

$$0 < z_1 = z_1^2 < z_2 = z_2^2 < \dots$$

(In fact, the Corollary of § 4.2 shows $z_i > 0$).

so,

$$u_n(x) = A_n \sin \sqrt{z_n} x$$

with z_n as given by graphical solution.

$$\text{Normalization : } \langle u_n, u_n \rangle = A_n^2 \int_0^1 |u_n|^2 \sqrt{2n} x \, dx = 1$$

$$\Rightarrow A_n^2 \frac{1}{2} \left(1 - \frac{\phi n 2\sqrt{2n}}{2\sqrt{2n}} \right) = 1$$

$$\Rightarrow u_n(x) = \frac{\sqrt{2}}{\left(1 - \frac{\phi n \sqrt{2n}}{2\sqrt{2n}} \right)^{1/2}} \text{ for } \sqrt{2n}x : e^{\text{'function}}$$

$$\phi n > 0 \quad \text{s.t.} \quad \tan \sqrt{2n} = -c \sqrt{2n} \quad : e^{\text{'values}}$$