

11/13/2017

(ii) $c=0$

as above and now BC at $x=1$ is

$$u(1) + \int_0^1 u'(1) dx = 0 \Rightarrow u(1) = 0$$

$$u_n(x) = A_n n \sqrt{a_n} x$$

$$u_n(1) = 0 \Rightarrow A_n n \sqrt{a_n} = 0 \Rightarrow n \sqrt{a_n} = 0$$

$$\text{u.s. } \sqrt{a_n} = n\pi, \quad n=1, 2, \dots \quad (\text{not } 2=0)$$

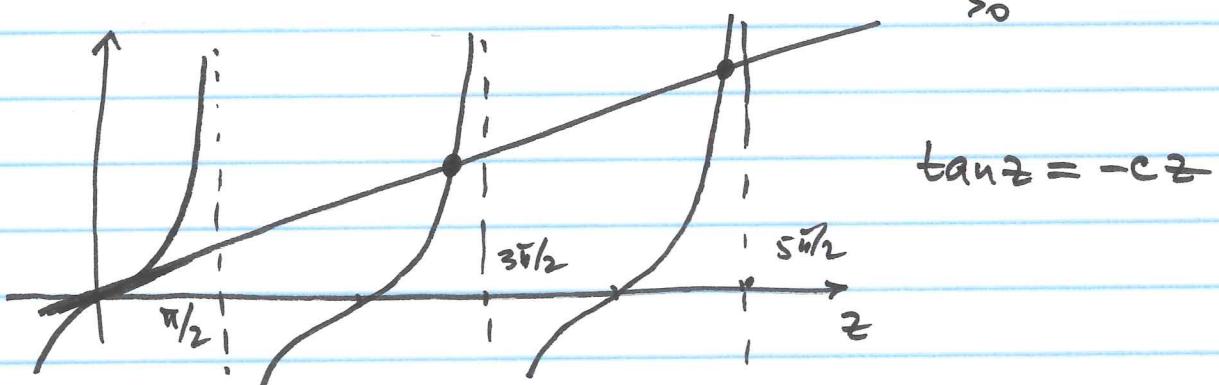
$$a_n = (n\pi)^2, \quad n=1, 2, \dots$$

Hence,

$$(a_n, u_n) = \{(n\pi)^2, \sqrt{2} \sin n\pi x\}, \quad n=1, 2, \dots$$

(iii) $c < 0$

$\tilde{z} > 0$ Set $\sqrt{z} = z$ again in $\tan \sqrt{z} = -c \sqrt{z}$



The roots are similar to those in (i), denote them by z_1, z_2, \dots giving $\tilde{z}_n = z_n^2$. Roots are all positive and

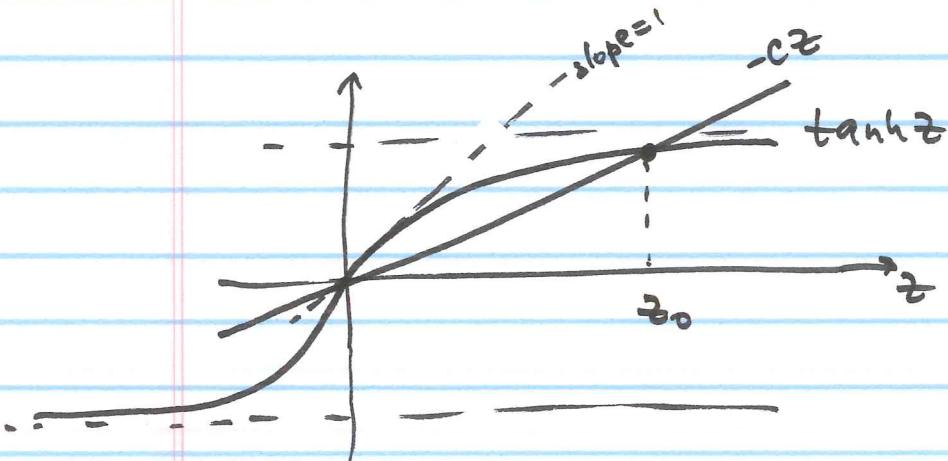
$$\tilde{z}_n \approx (2n+1) \frac{\pi}{2} - \frac{1}{c n^2} + \dots \quad \text{as } n \rightarrow \infty$$

asymptotic series

$\tilde{z}=0$ as in (i) $z=0$ is an eigenvalue iff $c=-1$

$z < 0$ Set $\sqrt{z} = i\bar{z}$, \bar{z} real, so

$$\tan i\bar{z} = -c i\bar{z} \quad \text{or} \quad \tanh \bar{z} = -c\bar{z}$$



We have one root if slope $-c$ is < 1 .
This root is \bar{z}_0 , which gives $\tilde{z}_0 = -\bar{z}_0^2 < 0$

w/ e' function

$$u_0 = A_0 \sinh \bar{z}_0 x$$

λ' values λ_n depend on c continuously.

$c > 0$ As c decreases, λ' values $0 < \lambda_1 < \lambda_2 < \dots$ are all positive and move to the right / increase.

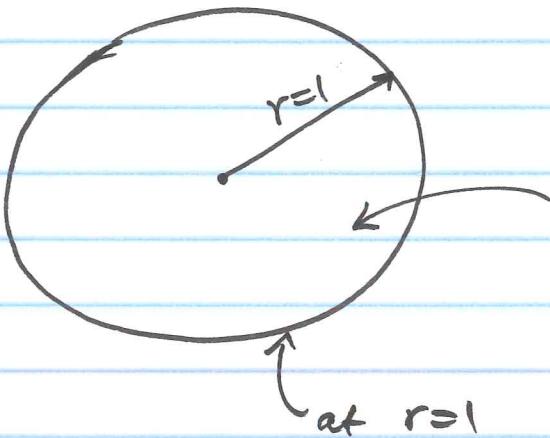
As c decreases to the critical value $c = -1$, a new λ' value $\lambda_0 > 0$ appears. As c decreases w/ $c < -1$, this λ' value moves to the left and gives $\lambda_0 < 0$ that decreases.

For $c < -1$

$$\theta(x, t) \sim a_0 v_0(x) e^{-\lambda_0 t} \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

since $\lambda_0 < 0$

2)



$$\nabla^2 v = \frac{1}{c^2} v_{tt}$$

$$v(1, t) = 0 \quad \forall t$$

$v(r, t)$ is bounded $\forall t$ and for $r \in [0, 1]$, in particular v has to be bounded at $r=0$, i.e.

$$|v(0, t)| < \infty$$

Initial condition: $v(r, 0) = f(r)$, $v_t(r, 0) = g(r)$

this gives $f(1) = 0$

$$g(1) = 0$$

In cylindrical coordinates (r, θ)

$$\nabla^2 v = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2}$$

Laplacian

For simplicity, we assume that the problem is axisymmetric, i.e. solution is independent of θ
 $(\Rightarrow \frac{\partial}{\partial \theta} = 0)$

$$\nabla^2 v = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \cancel{\frac{\partial^2 v}{\partial \theta^2}} = \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2}$$

Separable sol^u: $v(r, t) = u(r) F(t)$

$$F \frac{1}{r} (ru')' = \frac{u}{c^2} \ddot{F}$$

$$\Rightarrow \frac{1}{c^2} \frac{\ddot{F}}{F} = \frac{1}{ru} (ru')' = -2 = \text{const}$$

i.e. ..

$$\ddot{F} + 2c^2 F = 0 \quad (4.35)$$

$$(ru')' + 2ru = 0 \quad (4.36)$$

$$\text{BCs: } B_1 u = u(0) < \infty \quad (4.37)$$

$$B_2 u = u'(1) = 0$$

Note

1) (4.36) is SL eigenvalue problem w/ weight r , u .

$$\langle u, v \rangle = \int_0^1 uv r dr$$

$$2) (-pu')' + 2qw u = 0$$

$$p = -r \text{ and } w = r, \quad q = 1$$

Here p and w vanish at left endpoint $r=0$. Strictly speaking, this is a singular BVP but it has all the features of the regular BVP.

3) Consider eq^y (4.36):

$$(ru')' + 2ru = 0$$

$$u' + ru'' + 2ru = 0$$

$$\text{Let } z = \sqrt{r} r$$

$$u' = \frac{du}{dr} = \frac{du}{dz} \cdot \underbrace{\frac{dz}{dr}}_{\sqrt{r}} = \sqrt{z} \frac{du}{dz}$$

$$z^2 = 2r^2$$

$$u'' = z \frac{d^2u}{dz^2}$$

$$ru'' + u' + 2ru = 0$$

$$r^2 \frac{d^2u}{dz^2} + \sqrt{z} \frac{du}{dz} + 2 \cdot \frac{z}{\sqrt{z}} u = 0 \quad | \cdot r$$

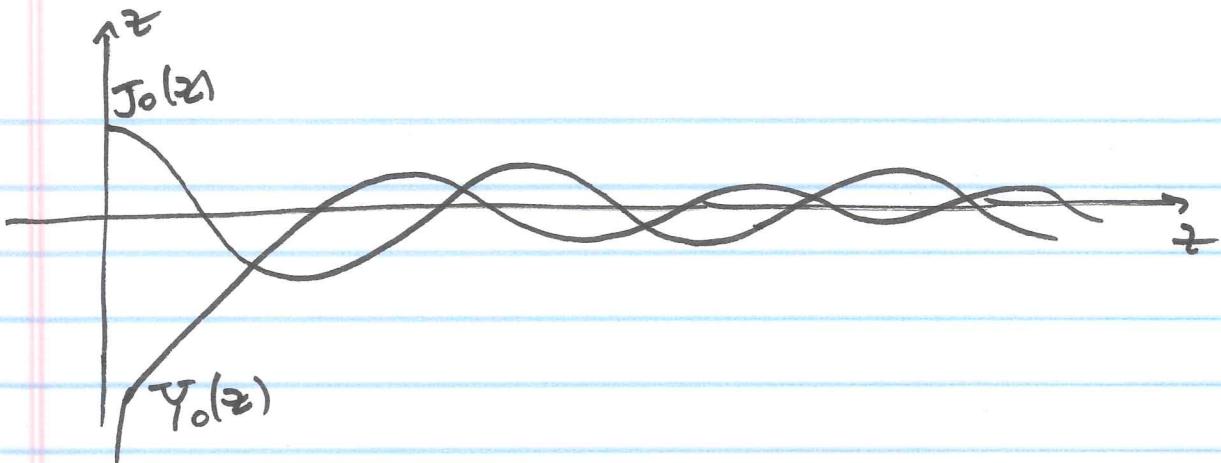
$$\frac{r^2 z}{z^2} \frac{d^2u}{dz^2} + \frac{r\sqrt{z}}{z} \frac{du}{dz} + \frac{\sqrt{z} \cdot r}{z^2} u = 0$$

$$\Rightarrow z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} + (z^2 - 0^2) u = 0 \quad \begin{matrix} \text{Bessel's} \\ \text{eq}^y \text{ of order 0} \end{matrix}$$

Solution:

$$u = A J_0(z) + B Y_0(z)$$

$J_0(z), Y_0(z)$: Bessel's functions of order 0
of 1st & second kind, respect.



$$|u(0)| < \infty \Rightarrow B = 0$$

$$z = \sqrt{2} r$$

$$\Rightarrow u(r) = A J_0(\sqrt{2}r)$$

$$u(r) \Big|_{r=1} = 0 \Rightarrow A J_0(\sqrt{2}) = 0$$

$\sqrt{2} = z_{0n} : n^{\text{th}} \text{ zeroes of } J_0(z)$