

11/15/2017

Aside:

(*)

$$x^2 \phi'' + x \phi' + 2\phi = 0$$

equipotential

equidimensional

or Euler eq²

$$\text{Let } \phi = x^l \Rightarrow \phi' = l x^{l-1}, \quad \phi'' = l(l-1) x^{l-2}$$

$$x^2 \cdot l(l-1) x^{l-2} + x l x^{l-1} + 2 x^l = 0$$

$$x^l [l(l-1) + l + 2] = 0$$

$$\Rightarrow l(l-1) + l + 2 = 0$$

$$l^2 - l + l + 2 = 0 \Rightarrow l^2 = -2$$

if $2 \neq 0 \Rightarrow l_{1,2} = \pm \sqrt{-2}$ that give two lin.

indep. solutions

x^{l_1} and x^{l_2}

if $2=0 \Rightarrow l^2=0 \Rightarrow l=0$ gives only one sol²

$$\phi = x^0 = 1$$

$$x^2 \phi'' + x \phi' + 2\phi = 0$$

$$x(x\phi'' + \phi') = 0$$

$$v = \phi'$$

$$xv' + v = 0 : \text{separable ODE}$$

$$xv' = -v$$

$$\frac{v'}{v} = -\frac{1}{x}$$

$$\frac{dv}{v} = -\frac{1}{x}$$

$$\ln v = -\ln x + \tilde{C} \Rightarrow v = \frac{c}{x}$$

" ϕ'

$$\Rightarrow \phi = \int \frac{c}{x} dx = c \ln x + \hat{C}$$

$\Rightarrow \ln x$ is another lin. indep. solution

Hence,

$$\phi = C_1 \cdot 1 + C_2 \ln x; \text{ general sol } \text{ of } (*)$$

$\phi = 1$: regular / well defined solution at $x=0$

$\phi = \ln x$: singular solution at $x=0$.

Back to Bessel's eq^u (4.36).

(4.36)

$$(ru')' + 2ru = 0$$

Bessel eq^u of
order 0

$$ru'' + u' + 2ru = 0 \quad | \quad r$$

$$r^2u'' + ru' + 2r^2u = 0$$

$$u'' + \frac{1}{r} u' + 2u = 0$$

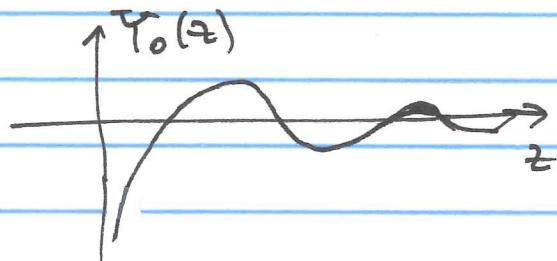
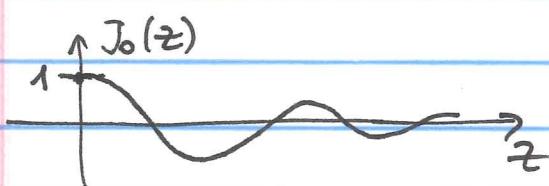
see Frobenius method
r=0 is a regular singular point, r=∞ is an irregular singular point

DES and
BVPs
by Edwards,
Penney
§ 8.3

As one can see, when $z=0$, we have equipotential equation. So, one solution of (4.36) is regular at $r=0$ and has a (Frobenius) power series solution. The second (lin. independent) sol^z is singular at $r=0$, like $\ln r$.

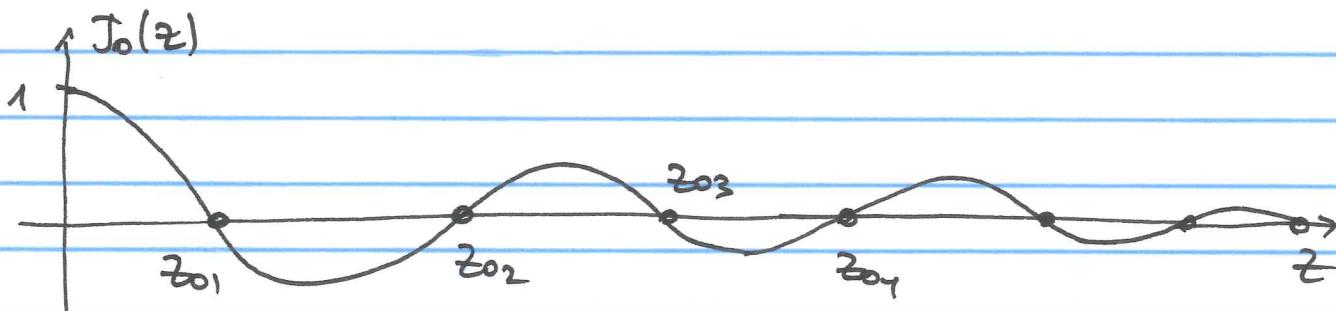
$$J_0(z) \sim 1 \quad \text{as } z \rightarrow 0 \quad z = \sqrt{2r}$$

$$Y_0(z) \sim \ln z \quad \text{as } z \rightarrow 0 \quad z = \sqrt{2r}$$



So, of two lin. independent solutions J_0 and Y_0 only $J_0(\sqrt{2r})$ satisfies the BC that is boundary at $r=0$, i.e.

$$u(r) = A J_0(\sqrt{2r})$$



$$\text{BC: } u(1) = 0 \Rightarrow A J_0(\sqrt{2}) = 0$$

x_0

$\Rightarrow \sqrt{2n} = z_{0n}$: zeros of $J_0(z)$
 z_{0n} : n^{th} zero of $J_0(z)$

$$z_n = z_{0n}^2 : e'\text{values}$$

$$z_{01} = 2.40483$$

$$z_{02} = 5.52008$$

$$z_{03} = 8.6573$$

$$z_{04} = 11.79153$$

- - - - -

The e' values are real, distinct and positive, and the e' functions are orthonormal

$$u_n(r) = A_n J_0(\sqrt{2n}r) : e'\text{functions}$$

$$\int_0^1 u_n^2(r) \cdot r dr = 1 \quad \text{or}$$

\oint
 weight

$$A_n^2 \int_0^1 J_0^2(\sqrt{2n}r) \cdot r dr = 1 \quad \text{gives } A_n$$

$$A_n^2 = \frac{1}{\int_0^1 J_0^2(\sqrt{2n}r) r dr}$$

Back to time-dependent eqⁿ

$$(4.35) \quad \ddot{F} + \frac{1}{c^2} F = 0$$

For each λ_n , (4.35) gives $F \sim e^{\pm i\sqrt{\lambda_n}t}$ or
 sin $\sqrt{\lambda_n}ct$ / cos $\sqrt{\lambda_n}ct$

The solution for v is a linear superposition

$$v(r, t) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}r) (a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t))$$

where a_n and b_n are determined by IC.

$$(+) \quad v(r, 0) = f(r) = \sum_{n=1}^{\infty} \underbrace{A_n J_0(\sqrt{\lambda_n}r)}_{a_n} a_n$$

$$\Rightarrow \boxed{a_n = \int_0^1 f(r) A_n J_0(\sqrt{\lambda_n}r) r dr}$$

$$(\mp) \quad v_t(r, 0) = g(r) = \sum_{n=1}^{\infty} \underbrace{A_n J_0(\sqrt{\lambda_n}r)}_{b_n \cdot c\sqrt{\lambda_n}} b_n$$

$$\Rightarrow \boxed{b_n = \frac{1}{c\sqrt{\lambda_n}} \int_0^1 g(r) A_n J_0(\sqrt{\lambda_n}r) r dr}$$

$(+)$ and (\mp) are Fourier-Bessel series.

Positive definite operators

Note the fact that e'values are all positive follow from the general result that:

The (real) e'values of a positive definite operator are positive.

Def The operator L is positive definite if $\langle u, Lu \rangle \geq 0$ for all $u \in D_B$.

Then if (λ, u) is an e'value, e'function pair and L is positive definite, then

$$0 \leq \langle u, Lu \rangle = \langle u, \lambda u \rangle = \lambda \langle u, u \rangle =$$

\uparrow
 since L is
 positive definite

$$= \lambda \|u\|^2 \quad (\lambda > 0 \text{ since } u \neq 0)$$

λ, u e'pair

u e'function $\Rightarrow \langle u, u \rangle > 0$

Ex $L = -\frac{1}{r} (ru')'$ is positive definite

$$B_2 u = u(1) \quad B_1 u = u(0)$$

□ $\langle u, Lu \rangle = \int_0^1 u \left(-\frac{1}{r}\right) (ru')' r dr =$

$$= - \int_0^1 u (ru')' dr \stackrel{\substack{\text{by} \\ \text{parts} \\ \text{once}}}{=} [ruu']_0^1 +$$

$$+ \int_0^1 r(u')^2 dr > 0$$

$u(1) = 0$
 $u(0)$ bounded
⇒ boundary term
is zero

So,

$$\langle u, Lu \rangle > 0 \text{ for } \forall u \in D_L$$