

4.6 Eigensystem of a regular non self-adjoint operator

$L \neq L^*$. We will look at the relation between the eigensystem (λ_n, u_n) of $\mathcal{L} = \{L, D_B\}$ and the eigensystem (λ'_n, v_n) of $\mathcal{L}^* = \{L^*, D_{B^*}\}$

Recall: (λ_n, u_n) satisfies $L u_n = \lambda_n u_n$,
 $u_n \in D_B$ $\underbrace{(B_i u_n = 0)}_5$

(λ'_n, v_n) satisfies $L^* v_n = \lambda'_n v_n$, $v_n \in D_{B^*}$
 $\underbrace{(B_i^* v_n = 0)}_3$

The Lagrange identity:

$$\langle v, Lu \rangle = \langle L^* v, u \rangle + [J(u, v)]$$

with $u \in D_B$, $v \in D_{B^*}$ gives

$$\langle v, Lu \rangle - \langle L^* v, u \rangle = 0 \quad (4.38)$$

since λ_n and u_n may be complex even for real L , the inner product now is

$$\langle u, v \rangle = \int_0^1 \bar{u} v w dx$$

where $\bar{-}$: complex conjugate

We need $\bar{-}$ to make sure that

$$\|u\|^2 = \langle u, u \rangle \text{ is real}$$

We also need

$$\begin{aligned} \langle \alpha u, v \rangle &= \int_0^1 \bar{\alpha} \bar{u} v w dx = \\ &= \bar{\alpha} \int_0^1 \bar{u} v w dx = \bar{\alpha} \langle u, v \rangle, \quad \text{ie.} \end{aligned}$$

$$\boxed{\langle \alpha u, v \rangle = \bar{\alpha} \langle u, v \rangle}$$

$$\boxed{\langle u, \alpha v \rangle = \alpha \langle u, v \rangle}$$

- 1) If L is real, meaning that all functions of x and parameters in L and b_i are real when x is real, then:

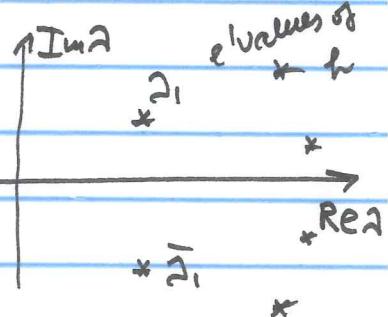
the eigenvalues and eigenfunctions of L and L^* occur in complex conjugate pairs.

Proof Let (λ, u) be an e'pair of L .

$$\Rightarrow Lu = \lambda u, \quad B_i u = 0 \quad \text{with } L \text{ and } B_i \text{ real}$$

$$\Rightarrow \overline{Lu} = \overline{\lambda u}, \quad \overline{B_i u} = 0 \quad \text{Z-plane}$$

$$L\bar{u} = \bar{\lambda}\bar{u}, \quad B_i \bar{u} = 0$$



Hence, if (λ, u) is an e'pair of L , so is $(\bar{\lambda}, \bar{u})$.
The same result holds for L^* . \blacksquare

2) If (λ_j, u_j) is an eigenpair of L and (λ'_i, v_i) is an eigenpair of L^* , then for

$$\bar{\lambda}_j \neq \bar{\lambda}'_i \quad (\text{different e'values})$$

$$\langle v_i, u_j \rangle = 0$$

(Orthogonality)

Proof

(λ_j, u_j) e'pair of L , (λ'_i, v_i) e'pair of L^* .

Then apply (4.38) with $u = u_j$, $v = v_i$:

$$0 = \langle v_i, L u_j \rangle - \langle L^* v_i, u_j \rangle = \\ \bar{\lambda}_j u_j - \bar{\lambda}'_i v_i$$

$$= \langle v_i, \bar{\gamma}_j u_j \rangle - \langle \bar{\gamma}_i' v_i, u_j \rangle =$$

$$= \bar{\gamma}_j \langle v_i, u_j \rangle - \bar{\gamma}_i' \langle v_i, u_j \rangle = \underbrace{(\bar{\gamma}_j - \bar{\gamma}_i')}_{\neq 0} \langle v_i, u_j \rangle$$

$$\Rightarrow \langle v_i, u_j \rangle = 0$$

3) If γ is an eigenvalue of L , then $\bar{\gamma}$ is an eigenvalue of L^* .

To prove this, we will use the following result:

For given L, L^* , parameter γ and continuous $f(x)$ EITHER

$$(i) \exists v \in D_{B^*} \text{ such that } (L^* - \gamma)v = f$$

OR

$$(ii) \exists v \in D_{B^*} \text{ such that } (L^* - \gamma)v = 0 \text{ for } v \neq 0$$

c. for given parameter γ EITHER

(i) the range of $L^* - \gamma$ is dense, the operator $L^* - \gamma$ is invertible and γ is

in the resolvent set (Resolvent: set of λ for which operator $L^* - \lambda$ is regular, i.e. it has bounded inverse $(L^* - \lambda)^{-1}$.)

OR

(ii) L^* has e-function v and λ is in the spectrum of L^* (i.e. λ is an eigenvalue of L^*)

Now, let's prove 3):

Proof

Suppose (λ, u) is an eigenpair of L , then

$$(L - \lambda)u = 0 \quad u \in D_B$$

For $\forall v \in D_{B^*}$

$$0 = \langle v, (L - \lambda)u \rangle = \langle v, Lu - \lambda u \rangle =$$

$$= \langle v, Lu \rangle - \lambda \langle v, u \rangle \stackrel{(4.38)}{=} \langle L^* v, u \rangle - \lambda \langle v, u \rangle$$

$$= \langle L^* v, u \rangle - \langle \bar{\lambda} v, u \rangle = \langle L^* v - \bar{\lambda} v, u \rangle =$$

$$= \langle (L^* - \bar{\lambda})v, u \rangle$$

Now suppose that $\bar{\lambda}$ is not an eigenvalue of L^* , i.e. $\bar{\lambda}$ is in the resolvent of L^* . Then we can choose an $f \neq 0$ such that

$$(L^* - \bar{\lambda})v = f \neq 0$$

$$\Rightarrow \text{RHS} = \langle (L^* - \bar{\lambda})v, u \rangle = \langle f, u \rangle \neq 0 \quad \downarrow$$

Since LHS = 0

$\Rightarrow \bar{\lambda} \notin \text{resolvent set} \Rightarrow \bar{\lambda} \in \text{spectrum of } L^*$,
i.e. $\exists v \in D_{B^*}$ such that $(L^* - \bar{\lambda})v = 0$.

Now we can discuss expansion of a function $f(x)$ in terms of eigenfunctions that form a complete set. If $f \in L^2(0, 1)$, then f has an eigenfunction expansion that converges in mean square.

Eigenfunctions of L :

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots ; u_1, u_2, \dots, u_n, \dots$$

$$\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n, \dots ; \bar{u}_1, \bar{u}_2, \dots, \bar{u}_n, \dots$$

System of \mathcal{L}^* :

$$\tilde{z}_1' = z_1, \tilde{z}_2' = z_2, \dots, \tilde{z}_n' = z_n, \dots; v_1, v_2, \dots, v_n, \dots$$

$$\bar{z}_1' = \bar{z}_1, \bar{z}_2' = \bar{z}_2, \dots, \bar{z}_n' = \bar{z}_n, \dots; \bar{v}_1, \bar{v}_2, \dots, \bar{v}_n, \dots$$

w/

$$\boxed{\langle v_i, v_j \rangle = \delta_{ij}}$$

In terms of e'functions of \mathcal{L} :

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \quad / \bar{v}_j$$

$$\underbrace{\int_0^1 \bar{v}_j f(x) w(x) dx}_{j \rightarrow n} = \underbrace{\sum_{n=1}^{\infty} a_n \int_0^1 \bar{v}_j u_n w dx}_{\langle \bar{v}_j, f \rangle} = a_j$$

$$\delta_{jn}$$

$$\Rightarrow \boxed{a_n = \langle v_n, f \rangle}$$

In terms of e'functions of \mathcal{L}^* :

$$f(x) = \sum_{n=1}^{\infty} b_n v_n(x)$$

$$\bar{f} = \sum_{n=1}^{\infty} \bar{b}_n \bar{v}_n \quad / \cdot u_j$$

$$\langle f, u_j \rangle = \int_0^1 \bar{f} u_j w dx = \sum_{n=1}^{\infty} \bar{b}_n \underbrace{\int_0^1 \bar{v}_n u_j w dx}_{\delta_{n,j}} = \bar{b}_j$$

$$\Rightarrow \boxed{\bar{b}_n = \langle f, u_n \rangle}$$

or $\boxed{b_n = \langle u_n, f \rangle}$

Since $\overline{\langle u, v \rangle} = \langle v, u \rangle$