

4.7 The relation between the Green's function and the eigensystem

Return to the regular SL BVP (with unmixed BCs):

$$(4.39) \quad \begin{aligned} (L - \mu w)u &= f & x \in (0,1) \\ B_i u &= c_i \end{aligned}$$

$$(n=2) \quad Lu = (-pu')' + qu$$

$$B_i u = \alpha_i u + \beta_i u'$$

where p and w are >0 for $x \in [0,1]$.
 The eigenvalue problem has a pure point spectrum (no continuous spectrum) with real eigenvalues λ_n and eigenfunctions $u_n(x)$.

In § 4.4 we saw that the eigenfunction expansion of the Green's function $G(x, \xi; \mu)$, which satisfies

$$(4.40) \quad \begin{aligned} (L - \mu w)G &= \delta(x - \xi) \\ B_i G &= 0 \end{aligned}$$

is related to the eigensystem by

$$(4.41) \quad G(x, \xi; \mu) = - \sum_{n=1}^{\infty} \frac{u_n(x) u_n(\xi)}{\mu - \lambda_n}$$

If the eigensystem $(\lambda_n, u_n(x))$ is known then $G(x, \xi; \mu)$ can be constructed from it.

Conversely, if we happen to know G in terms of elementary functions, can we find the eigensystem from it? (without solving the eigenvalue problem!)

Yes, we can use (4.41) — where we know that the sum converges absolutely and uniformly (Mercer's Thm).

Consider G as a function of a complex parameter μ .

Note Review Laurent series, poles, residues, residue Thm

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n =$$

$$= \sum_{n=-\infty}^{-1} C_n (z-a)^n + \sum_{n=0}^{\infty} C_n (z-a)^n =$$

$$\begin{aligned}
 &= \dots + C_{-3} \frac{1}{(z-a)^3} + C_{-2} \frac{1}{(z-a)^2} + \frac{C_{-1}}{z-a} + C_0 + \\
 &\quad + C_1(z-a) + C_2(z-a)^2 + \dots
 \end{aligned}$$

$$\text{Res}_{a^+} f = C_{-1}$$

1) G has simple poles at $\mu = \lambda_n$. So, locating the poles from the closed form expression for G determines the eigenvalues λ_n .

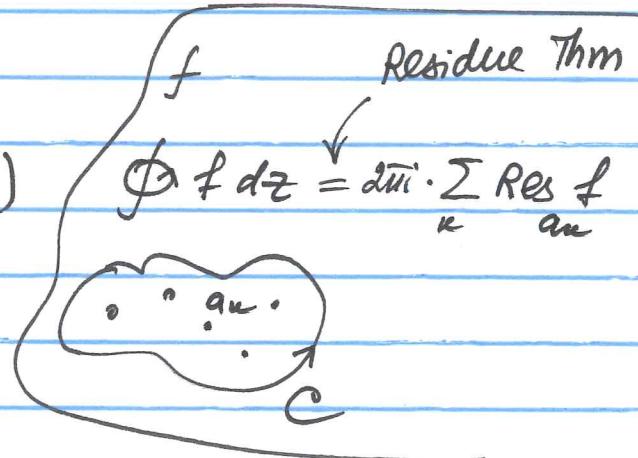
2) The residue at $\mu = \lambda_n$ is $-u_n(x)u_n(\xi)$. So, computing the residue of G at each pole from the closed form expression of G , determines $u_n(x)$.

Integrating (4.41) around the large circle C_∞ in the μ -plane, we get an expression:

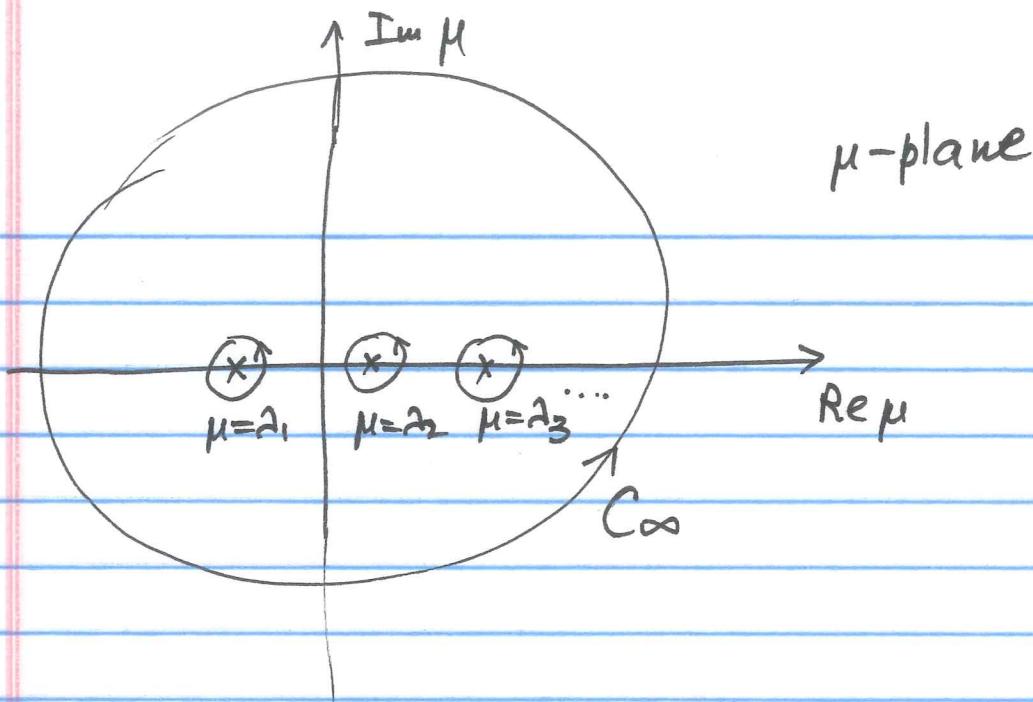
(4.42)

$$\frac{1}{2\pi i} \oint_{C_\infty} G(x, \xi; \mu) d\mu = - \sum_{n=1}^{\infty} u_n(x) u_n(\xi)$$

$\underbrace{}$
 $= \sum_{n=1}^{\infty} \text{Res } G$



via the residue Thm.



Recall the expression of $\delta(x - \xi)$ from §4.3:

$$\delta(x - \xi) = w(x) \sum_{n=1}^{\infty} u_n(x) u_n(\xi)$$

Thus (4.42) can also be written

$$(4.43) \quad \frac{1}{2\pi i} \oint_{C_\infty} G(x, \xi; \mu) d\mu = - \frac{\delta(x - \xi)}{w(x)}$$

so, (4.42) is formal since the series on the RHS does not converge (to a function but to a distribution $\delta(x - \xi)$).

Ex

$$L = -\frac{d^2}{dx^2} \quad x \in (0, 1)$$

$$B_1 u = u(0)$$

$$B_2 u = u(1)$$

$$w(x) = 1$$

For this example we have both the eigensystem and a closed form expression for the Green's function.

Eigensystem:

$$(A_n, u_n) = \left(\left(\frac{n\pi}{l} \right)^2, \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l} \right), n=1, 2, \dots$$

as found in example 1, section 4.1.

The problem for G :

$$-G'' - \mu G = \delta(x - \xi)$$

$$G(0) = G(1) = 0$$

has the solution

$$G(x, \xi; \mu) = \frac{\sin \sqrt{\mu} \cdot x_{<} \cdot \sin \sqrt{\mu} (x - x_{>})}{\sqrt{\mu} \cdot \sin \sqrt{\mu} l} \quad 0 \leq x, \xi \leq l$$

HW: check

$$x_{<} = \min(x, \xi)$$

$$x_{>} = \max(x, \xi)$$

$$\mu \in \mathbb{C}.$$

As a function of μ , G has simple poles at $\sqrt{\mu}l = n\bar{u}$, $n \in \mathbb{Z}$, i.e. at

$$\mu = \mu_n = \left(\frac{n\bar{u}}{l}\right)^2, \quad n=1, 2, \dots$$

and we can check separately that $\mu=0$ is NOT a pole:

$$\left(\lim_{\mu \rightarrow 0} G(x, \xi; \mu) = \frac{x_< (l - x_>)}{l} \right)$$

These are in fact eigenvalues of L .

The residue of G at $\mu_n = \left(\frac{n\bar{u}}{l}\right)^2$ is

Aside

$$f(z) = \frac{g(z)}{h(z)} \quad \text{f has a simple pole at } z=a \\ \text{i.e. } g(a) \neq 0, \quad h(a) = 0$$

then

$$\operatorname{Res}_a f = \frac{g(a)}{h'(a)}$$

Cheek

$$\operatorname{Res}_{\mu_n} G = \sin \frac{n\bar{u}}{l} x_< \cdot \sin \frac{n\bar{u}}{l} (l - x_>) \cdot \frac{l}{n\bar{u}} (-1)^n \frac{\partial \frac{n\bar{u}}{l}}{\partial^2} = \\ (-1)^{n+1} \sin \frac{n\bar{u} x_>}{l}$$

$$= -\frac{2}{\ell} \sin \frac{n\pi x <}{\ell} \cdot \sin \frac{n\pi x >}{\ell} = -u_n(x) u_n(\xi)$$

$$\text{so, } u_n(x) = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi x}{\ell}$$

these are eigenfunctions of L .