

11/29/2017

No class this Friday

BUT

There is a make-up class tomorrow
Thursday, Nov. 30

12:30 - 1:20 pm

TLC 248

4.10 APPROXIMATION OF EIGENVALUES USING VARIATIONAL METHODS

For an arbitrary 2nd order differential operator + BCs, L , we can find λ 'values as the roots of transcendental equation, provided we know two linearly independent solutions of the homogeneous ODE $(L - \lambda w)u = 0$ in closed form.

These solutions are often not known in closed form or only known approximately and then we need a method for approximating the λ 'values. Of these, it is often the smallest λ 'value that is important in applications.

Here, we approximate the λ 'values of the Sturm-Liouville differential operator L .

$$(4.62) \quad Lu = (-pu')' + qu = \lambda w u \quad x \in (0, 1)$$

with $B_i u = 0 \quad i=1,2$

(i) UNMIXED BCs - regular SL eigenvalue problem

(ii) PERIODIC BCs - with $p(x)$ periodic

and use the space of functions

$$u \in D_B = \{ u : u \text{ pw } C^1(0,1) \text{ and } B_i u = 0 \}$$

Def The Rayleigh Quotient for (4.26) is

$$(4.63) \quad p(u) = \frac{\int_0^1 u L u dx}{\int_0^1 u^2 w dx} \quad \text{no } w$$

It is a functional which maps each function $u \in D_B$ to a scalar.

It is clear that if $p(u)$ is evaluated on an e'function u_n ($u_n \in D_B$), then it gives

$$p(u_n) = \lambda_n$$

where λ_n is the corresponding eigenvalue

$$\therefore u = u_n \Rightarrow L u_n = \lambda_n w u_n$$

\Rightarrow numerator in (4.63) is

$$\int_0^1 u_n L u_n dx = \int_0^1 u_n \lambda_n w u_n dx =$$

$$= \lambda_n \int_0^1 u_n^2 w dx$$

i.e.

$$g(u_n) = \frac{\int_0^1 u_n L u_n dx}{\int_0^1 u_n^2 w dx} = \frac{\int_0^1 u_n \cdot \lambda_n w u_n dx}{\int_0^1 u_n^2 w dx} =$$

$$= \frac{\lambda_n \int_0^1 u_n^2 w dx}{\int_0^1 u_n^2 w dx} = \lambda_n$$

$\underbrace{\quad}_{\neq 0}$
 u_n : eigenfunction

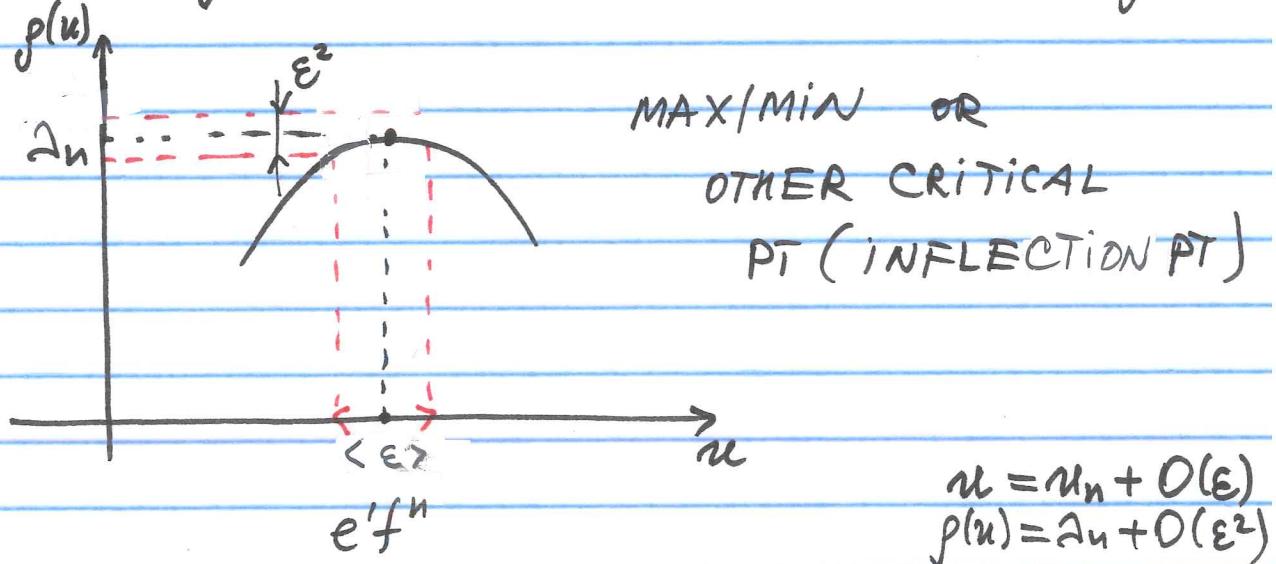
The fact that $g(u)$ gives a useful means for approximating eigenvalues follows from

LEMMA

For function $u \in D_B$, $\rho(u)$ is STATIONARY about $\rho(u) = \bar{u}_n$

$\Leftrightarrow u = u_n$ is the e' function of (4.62) with e' value \bar{u}_n .

Def For function $u \in D_B$, function $\rho(u)$ is called STATIONARY if change of order $O(\varepsilon)$ in argument u gives the change of order $O(\varepsilon^2)$ in values $\rho(u)$.



First order approximation of e' function gives second order approximation of e' value.

Proof

Consider $p(u)$ with $u = \phi + \varepsilon\psi$
where ϕ, ψ and, hence, u are any
functions in D_B

$$p(u) = \frac{\int_0^1 u L u dx}{\int_0^1 u^2 w dx} =$$

$$= \frac{\int_0^1 (\phi + \varepsilon\psi) L (\phi + \varepsilon\psi) dx}{\int_0^1 (\phi + \varepsilon\psi)^2 w dx} =$$

$$= \lambda + \frac{\int_0^1 (\phi + \varepsilon\psi) (L - 2w)(\phi + \varepsilon\psi) dx}{\int_0^1 (\phi + \varepsilon\psi)^2 w dx} \quad (T)$$

identity true

for $\forall \lambda$

\therefore

(*)

$$\textcircled{*} = \frac{\int_0^1 (\phi + \varepsilon\psi) L (\phi + \varepsilon\psi) + (\phi + \varepsilon\psi) (-2w)(\phi + \varepsilon\psi) dx}{\int_0^1 (\phi + \varepsilon\psi)^2 w dx}$$

$$= \frac{\int_0^1 (\phi + \varepsilon\psi) L (\phi + \varepsilon\psi) dx}{\int_0^1 (\phi + \varepsilon\psi)^2 dx} - \lambda \frac{\int_0^1 (\phi + \varepsilon\psi)^2 w dx}{\int_0^1 (\phi + \varepsilon\psi)^2 w dx}$$

The numerator in (T) is

$$\int_0^1 (\phi + \varepsilon \psi)(L - 2\omega)(\phi + \varepsilon \psi) dx =$$

$$= \int_0^1 \phi(L - 2\omega)\phi dx + \varepsilon \int_0^1 \phi(L - 2\omega)\psi + \psi(L - 2\omega)\phi dx$$

$$+ \varepsilon^2 \int_0^1 \psi(L - 2\omega)\psi dx$$

so, the term of order ε is

$$(+) \quad \int_0^1 \phi(L - 2\omega)\psi + \psi(L - 2\omega)\phi dx$$

and as a power series in ε we have

$$p(u) = \frac{\int_0^1 \phi L \phi dx}{\int_0^1 \omega \phi^2 dx} + \varepsilon \cdot \frac{2}{\int_0^1 \omega \phi^2 dx} *$$

$$* \left(\int_0^1 \omega \phi^2 dx \cdot \int_0^1 \psi(L - 2\omega)\phi dx - \right.$$

$$-\int_0^1 \omega \phi \psi dx \cdot \int_0^1 \phi(L-2\omega) \phi dx \Big) + O(\varepsilon^2)$$

Indeed,

$$\begin{aligned} p(u) &= 2 + \frac{\int_0^1 (\phi + \varepsilon \psi)(L-2\omega)(\phi + \varepsilon \psi) dx}{\int_0^1 \omega(\phi + \varepsilon \psi)^2 dx} = \\ &= 2 + \frac{\int_0^1 \phi(L-2\omega)\phi dx + \varepsilon \int_0^1 \psi(L-2\omega)\phi + \phi(L-2\omega)\psi dx + \varepsilon^2 \int_0^1 \psi(L-2\omega)\psi dx}{\int_0^1 \omega(\phi^2 + 2\varepsilon \phi \psi + \varepsilon^2 \psi^2) dx} \end{aligned}$$

$$= 2 + \frac{1}{\int_0^1 \omega \phi^2 dx + 2\varepsilon \int_0^1 \phi \psi \omega dx + \varepsilon^2 \int_0^1 \omega \psi^2 dx} *$$

$$\begin{aligned} &+ \left(\int_0^1 \phi(L-2\omega)\phi dx + \varepsilon \int_0^1 \psi(L-2\omega)\phi + \phi(L-2\omega)\psi dx + \right. \\ &\quad \left. + \varepsilon^2 \int_0^1 \psi(L-2\omega)\psi dx \right) \equiv \end{aligned}$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots : \text{geometric series}$$

(provided $|x| < 1$)

Hence

$$\frac{1}{\int_0^1 \omega \phi^2 dx + 2\varepsilon \int_0^1 \phi \psi \omega dx + \varepsilon^2 \int_0^1 \omega \psi^2 dx} =$$

$$= \frac{1}{\int_0^1 \omega \phi^2 dx \left[1 + \frac{2\varepsilon \int_0^1 \phi \psi \omega dx + \varepsilon^2 \int_0^1 \omega \psi^2 dx}{\int_0^1 \omega \phi^2 dx} \right]} =$$

$$= \frac{1}{\int_0^1 \omega \phi^2 dx} \cdot \left(1 - 2\varepsilon \frac{\int_0^1 \phi \psi \omega dx}{\int_0^1 \omega \phi^2 dx} + O(\varepsilon^2) \right)$$

Therefore,

$$\begin{aligned} &\equiv \lambda + \frac{1}{\int_0^1 \omega \phi^2 dx} \left(1 - 2\varepsilon \frac{\int_0^1 \phi \psi \omega dx}{\int_0^1 \omega \phi^2 dx} + O(\varepsilon^2) \right) * \\ &* \left(\int_0^1 \phi(L-\lambda\omega)\phi dx + \varepsilon \int_0^1 \psi(L-\lambda\omega)\phi + \phi(L-\lambda\omega)\psi dx + \right. \end{aligned}$$

$$+ \varepsilon^2 \int_0^1 \psi(L - 2\omega) \phi dx \Big) =$$

$$= 2 + \frac{\int_0^1 \phi(L - 2\omega) \phi dx}{\int_0^1 \omega \phi^2 dx} +$$

$$+ \varepsilon \cdot \frac{1}{\int_0^1 \omega \phi^2 dx} \cdot \left\{ \int_0^1 \psi(L - 2\omega) \phi + \phi(L - 2\omega) \psi dx - \right.$$

$$\left. - 2 \frac{\int_0^1 \phi \psi \omega dx \cdot \int_0^1 \phi(L - 2\omega) \phi dx}{\int_0^1 \omega \phi^2 dx} \right\} + O(\varepsilon^2) \quad \square$$

By Lagrange identity, since $\phi, \psi \in D_B$,

$J(\phi, \psi) = 0$ and $L = L^*$, we have

$$\int_0^1 \phi L \psi dx = \int_0^1 \psi L \phi dx$$

and the term

$$** = \int_0^1 \psi(L - 2\omega) \phi + \phi(L - 2\omega) \psi dx \equiv$$

simplifies to

$$\begin{aligned} \textcircled{=} \int_0^1 \psi L \phi - \psi 2\omega \phi + \phi L \psi - \phi 2\omega \psi dx &= \dots \\ &= 2 \int_0^1 \psi L \phi - 2\omega \psi \phi dx = 2 \int_0^1 \psi (L - 2\omega) \phi dx \end{aligned}$$

Hence,

$$\begin{aligned} \boxed{\Xi} \quad & 2 + \frac{\int_0^1 \phi (L - 2\omega) \phi dx}{\int_0^1 \omega \phi^2 dx} + \\ & + \varepsilon \frac{1}{\int_0^1 \omega \phi^2 dx} \left\{ 2 \int_0^1 \psi (L - 2\omega) \phi dx - 2 \frac{\int_0^1 \phi \psi \omega dx \cdot \int_0^1 \phi (L - 2\omega) \phi dx}{\int_0^1 \omega \phi^2 dx} \right\} \\ & + O(\varepsilon^2) = \frac{\int_0^1 \phi L \phi dx}{\int_0^1 \omega \phi^2 dx} + \\ & + \varepsilon \frac{1}{\int_0^1 \omega \phi^2 dx} \cdot \frac{2}{\int_0^1 \omega \phi^2 dx} \left\{ \int_0^1 \psi (L - 2\omega) \phi dx \cdot \int_0^1 \omega \phi^2 dx - \right. \\ & \left. - \int_0^1 \phi \psi \omega dx \cdot \int_0^1 \phi (L - 2\omega) \phi dx \right\} + O(\varepsilon^2) \end{aligned}$$

or

$$f(u) = \frac{\int_0^1 \phi L \phi dx}{\int_0^1 w \phi^2 dx} + \frac{2\epsilon}{\left(\int_0^1 w \phi^2 dx\right)^2} *$$

$$* \left\{ \int_0^1 \psi(L-2w) \phi dx \cdot \int_0^1 w \phi^2 dx - \int_0^1 \phi \psi w dx \cdot \int_0^1 \phi(L-2w) \phi dx \right\}$$

$$+ O(\epsilon^2)$$