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$\mathcal{L} = \{L, D_B\}$  and the adjoint of  $\mathcal{L}$

Def The operator to BVP,  $\mathcal{L}$ , consists of the differential operator

$$L = \sum_{j=0}^n a_j(x) \frac{d^{n-j}}{dx^{n-j}}$$

and its domain  $D_B$ .

For  $L$ , consider  $x \in (a, b)$  to be a finite interval. Then we can rescale it so that  $x \in (0, 1)$ , and we take  $a_j(x) = C^n(0, 1)$

Domain  $D_B$  is the set of functions on which we let  $L$  act. If function  $u \in D_B$ , then

(i)  $u$  is smooth enough, here  $u \in C^n(0, 1)$

(ii)  $u$  satisfies homogeneous BCs  $\downarrow$   
 $Lu \in C^0(0, 1)$

$$\beta_i u = 0, \quad i = 1, \dots, n$$

(where also  $(\alpha, \beta)$  has rank  $n$ , so that BCs are independent and BVP is well-posed.)

so,

$$u \in D_B \Rightarrow u \in C^n(0,1), \quad B_i u = 0, \quad i=1, \dots, n$$

The need to use the inner product also means that  $u$  and  $Lu$  need to be square integrable:

$$\int_0^1 u^2 dx < \infty, \quad \int_0^1 (Lu)^2 dx < \infty$$

i.e.  $u, Lu \in L^2(0,1)$ .

Since  $u$  and  $Lu$  are continuous on the finite interval, they are automatically square integrable. In case our domain is, say,  $x \in (-\infty, \infty)$ , then square integrability should be required.

The inner product  $\langle u, v \rangle = \int_0^1 uv w dx$   
 induces a norm on weight functions:

$$\|u\| = \sqrt{\langle u, u \rangle} = \left( \int_0^1 u^2 w dx \right)^{1/2}$$

These conditions imply that domain  $D_B$  is a linear function space with an inner product. It may not be a

Hilbert space because  $D_B$  may not be complete, i.e. there may exist a sequence  $u_n \in D_B$  with the limit that is not in  $D_B$ .

Def The adjoint operator of  $L$  (diff. operator + BCs) is

$$L^* = \{ L^*, D_B^* \}$$

Here  $L^*$  is the formal adjoint of  $L$ . We know how to construct it.

$D_B^*$  is the "dual" of  $D_B$ , function

$u \in D_B^*$  if

$$(i) \quad u \in C^n(0,1)$$

$$(ii) \quad B_i^* u = 0 \quad \begin{matrix} \Xi \\ i=1, \dots, n \end{matrix}$$

$B_i^* u = 0$ : adjoint homogeneous BCs.

Def The adjoint homogeneous boundary conditions  $B_i^* v = 0$ ,  $i=1, \dots, n$ , are such that

$$[J(u, v)]_0^1 = 0 \quad \text{for all arbitrary functions } u \in D_B$$

$B_i^*$  are constructed by letting  $[J(u, 0)]_0' = 0$  and using  $B_i u = \frac{1}{\xi}$  for arbitrary functions  $u \in D_B$ .

It turns out that  $B_i^*$  are not unique, but non-uniqueness does not matter.

Ex 1 .

$$L = \frac{d^2}{dx^2} + x \frac{d}{dx} + e^x \quad x \in (0, 1)$$

$$B_1 u = u(0) + u(1)$$

$$B_2 u = u'(0) + u'(1)$$

Note: 2-point BVP in general is given as

$$\begin{aligned} Lu &= f \\ B_i u &= c_i \end{aligned}$$

are given

$f$  and  $c_i$  are not required for construction of  $L^*$  and  $B_i^*$ .

$$D_B = \{u \in C^2[0, 1], \quad B_1 u = B_2 u = 0\}$$

$$\langle f, g \rangle = \int_0^1 f g dx$$

Consider

$$\langle v, Lu \rangle = \int_0^1 v(u'' + xu' + e^x u) dx \quad \text{by parts}$$

$v \in C^2(0, 1)$ , arbitrary

$u \in D_B$ , arbitrary

$$\begin{aligned} & \Rightarrow [vu' + xuv - uv']_0^1 + \int_0^1 u(v'' - (xv)' + e^x v) dx \\ & \qquad \qquad \qquad J(u, v) \qquad \qquad \qquad \langle L^* v, u \rangle \end{aligned}$$

$$L^* = \frac{d^2}{dx^2} - \frac{d}{dx}(x \cdot) + e^x$$

$L^* \neq L \Rightarrow L$  is not formally self-adjoint

$\Rightarrow L$  is not self-adjoint

To find  $B_i^*$ , set  $[J(u, v)]_0^1 = 0 =$

$$= v(1)u'(1) + u(1)v(1) - u(1)v'(1) - v(0)u'(0) +$$

$$+ u(0)v'(0) \quad \textcircled{=} \quad - u'(1)$$

$$\underline{\underline{-u(1)}}$$

We want this to be 0 for arbitrary  $u$ :

$$B_1 u = 0 \Rightarrow u(0) + u(1) = 0 \Rightarrow u(1) = -u(0)$$

$$B_2 u = 0 \Rightarrow u'(0) + u'(1) = 0 \Rightarrow u'(1) = -u'(0)$$

$$\Leftrightarrow u'(1) (v(1) + v(0)) + u(1) (v(1) - v'(1) - v'(0))$$

Since  $u(1)$  and  $u'(1)$  are arbitrary, for the entire expression to be zero, (...) and (...) have to be zero, i.e.

$$v(1) + v(0) = 0$$

$$v(1) - v'(1) - v'(0) = 0$$

so we define

$$B_1^* v = v(1) + v(0) = 0$$

$$B_2^* v = v(1) - v'(1) - v'(0) = 0$$

Notice

$$B_1^* = B_1$$

$$B_2^* \neq B_2$$

The adjoint problem is

$$L^* = \frac{d^2}{dx^2} - \frac{d}{dx}(x \cdot) + e^x, \quad x \in (0, 1)$$

$$B_1^* v = v(0) + v(1)$$

*boundary operators*

$$-B_2^* v = v(1) - v'(1) - v''(0)$$

Adjoint homogeneous problem is

$$L^* v = 0, \quad x \in (0, 1)$$

$$B_i^* v = 0, \quad i=1, 2$$

$$\langle f, g \rangle = \int_0^1 f g dx$$

Ex 2

$$L = e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx} + 1 = \frac{d}{dx} (e^x \frac{d}{dx}) + 1,$$

$$x \in (0, 1)$$

Notice here  $L = L^*$  straight away since  $L$  has the form

$$\frac{d}{dx} (a_0 \frac{d}{dx}) + a_2$$

(see prev.  
lecture)

$$a_0 = a_0(x) = e^x, \quad a_2 = a_2(x) = 1$$

i.e.  $L$  is formally self-adjoint.

$$B_1 u = u'(0)$$

$$\langle f, g \rangle = \int_0^1 f g \, dx$$

$$B_2 u = u(0) + u(1)$$

$$D_B = \{ u : u \in C^2(0,1) \text{ and } B_i u = 0 \}$$

Consider

$$\langle v, Lu \rangle \circledcirc$$

$v \in C^2(0,1)$ , arbitrary

$u \in D_B$ , arbitrary

$$\begin{aligned} \circledcirc & \int_0^1 v ((e^x u')' + u) \, dx = \underset{\text{parts}}{\underset{|}{\text{by}}} \left[ e^x (u'v - uv') \right]_0^1 \\ & + \int_0^1 u ((e^x v')' + v) \, dx \end{aligned}$$

$$\langle L^* v, u \rangle$$

$$L^* = \frac{d}{dx} \left( e^x \frac{d}{dx} \cdot \right) + 1 \Rightarrow L^* = L.$$

So,  $L$  is formally self-adjoint.