

FORMULATION OF THE RAYLEIGH-RITZ METHOD BY LAGRANGE MULTIPLIERS

For a single eigenvalue and eigenfunction, we can ensure that the eigenfunction is normalized by finding stationary values of the Rayleigh quotient

$$R(u) = \frac{\int_0^1 u L u dx}{\int_0^1 w u^2 dx}$$

for which the denominator is 1.

This is a constraint extremum problem; it is equivalent to finding stationary values of the numerator

$$I(u) = \int_0^1 u L u dx$$

subject to the constraint (normalization of u) that the denominator

$$Q(u) = \int_0^1 w u^2 dx = 1$$

Now we can apply the method of Lagrange multipliers and look for functions $u \in D_B$ which make

$$F(u) = I(u) - \lambda Q(u)$$

Lagrange multiplier

stationary among arbitrary functions $u \in D_B$ subject to the constraint

$$Q(u) = 1$$

Sometimes we write $I(u)$ in an equivalent form to $\int u L u dx$ by applying one integration by parts. Here, as earlier

$$L u = -(pu')' + g u$$

Numerator

$$I(u) = \int_0^1 u L u dx = \int_0^1 u (- (pu')' + g u) dx =$$

$$U = u$$

$$\begin{aligned} dU = u' dx &\stackrel{\text{by parts}}{=} \int_0^1 (pu')^2 + g u^2 dx - p(1) u(1) u'(1) + \\ dV = (pu')' dx & \quad + p(0) u(0) u'(0) \end{aligned}$$

for unmixed BG

$$\beta_1 u = \alpha_1 u(0) + \beta_1 u'(0) = 0 \Rightarrow u(0) = -\frac{\beta_1}{\alpha_1} u'(0)$$

$$\beta_2 u = \alpha_2 u(1) + \beta_2 u'(1) = 0 \Rightarrow u(1) = -\frac{\beta_2}{\alpha_2} u'(1)$$

can be written as

$$\alpha_1, \alpha_2 \neq 0$$

or

$$u'(0) = -\frac{\alpha_1}{\beta_1} u(0)$$

$$u'(1) = -\frac{\alpha_2}{\beta_2} u(1)$$

$$\beta_1, \beta_2 \neq 0$$

$$I(u) = \int_0^1 \left(p(u')^2 + q u^2 \right) dx + \begin{cases} p(1) \frac{\alpha_2}{\beta_2} u^2(1) - p(0) \frac{\alpha_1}{\beta_1} u^2(0) \\ p(1) \frac{\beta_2}{\alpha_2} (u'(1))^2 - p(0) \frac{\beta_1}{\alpha_1} (u'(0))^2 \end{cases}$$

Ex Approximate the lowest ϵ 'value and
= ϵ function of the ϵ 'value problem

$$u'' + 2u = 0$$

$$u(0) = 0, \quad u(1) = 0$$

(We already know

$$\lambda_n = (n\pi)^2$$

$n=1, 2, \dots$

$$u_n = \sqrt{2} \sin n\pi x$$

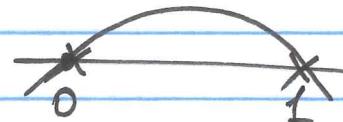
so, the smallest eigenvalue is

$$\lambda_1 = \pi^2 \quad \text{and} \quad u_1 = \sqrt{2} \sin \pi x : \text{EXACT}$$

We choose a space of "trial" functions $u \in D_B$. Polynomials are a suitable/common choice and the more "similar" the trial function is to the exact function, the better the approximation is.

$$u \in D_B \Rightarrow u=0 \text{ at } x=0 \text{ and } x=1$$

The linear function is not good \Rightarrow choose a parabola



Note Since we have the lowest eigenvalue \Rightarrow no zeros inside $[0, 1]$

$n=1 \Rightarrow$ zero roots in $(0, 1)$

Note A function of order n has exactly $n-1$ roots inside $[0, 1]$.

Hence, choose a polynomial of degree 2.

$$u(x) = Ax^2 + Bx + C^0$$

$$u(0)=0 \Rightarrow C=0$$

$$u(1)=A+B=0 \Rightarrow B=-A$$

\Rightarrow

$$u(x) = Ax(x-1)$$

or $\boxed{u(x) = Ax(1-x)}$

$$u(x) = Ax - Ax^2$$

$$u' = A - 2Ax, \quad u'' = -2A$$

will use this form

A: parameter to be found

Note since the stationary value of $p(u) = 2$,
an e-value and $p(u) = \frac{I(u)}{Q(u)}$ where $Q(u)=1$,

we have $I(u) = 2$, e-value.

Then

$$u'' + 2u = 0 \Rightarrow Lu = -u''$$

$$I(u) = \int_0^1 u Lu dx = \int_0^1 u(-u'') dx =$$

$$= 2A^2 \int_0^1 x(1-x) dx = \frac{A^2}{3}$$

$$Q(u) = \int_0^1 u^2 dx = A^2 \int_0^1 x^2(1-x)^2 dx = \frac{A^2}{30}$$

Method of Lagrange multipliers: we set

$$\mathcal{F}(u) = I(u) - \lambda(Q(u) - 1)$$

so

$$\mathcal{F}(u) = \frac{A^2}{3} - \lambda \left(\frac{A^2}{30} - 1 \right)$$

This is to be stationary wrt A (for u)
and λ (for constraint)

$$\frac{\partial \mathcal{F}}{\partial A} = \frac{2A}{3} - \lambda \cdot \frac{2A}{30} = 0$$

$$\frac{2A}{3} \left(1 - \frac{\lambda}{10} \right) = 0$$

$\cancel{\times 0}$

$$A \neq 0 \text{ since } u \text{ is e'function} \Rightarrow 1 - \frac{\lambda}{10} = 0$$

$$\Rightarrow \lambda = 10$$

In addition,

$$\frac{\partial \mathcal{F}}{\partial \lambda} = \frac{A^2}{30} - 1 = 0 \Rightarrow A = \sqrt{30} \quad \begin{matrix} \\ \text{wlog} \end{matrix}$$

Then

$$I(u) = 10$$

is an approximation to \bar{z}_1 with

$$u = \sqrt{30}x(1-x)$$

as an approximation to u_1

Error in \bar{z}_1 is

$$\text{perc. error} = \frac{|\bar{z}_1 - z_1|}{\bar{z}_1} \cdot 100\% \approx 1.3\%$$

$\underbrace{}_{\text{exact}}$

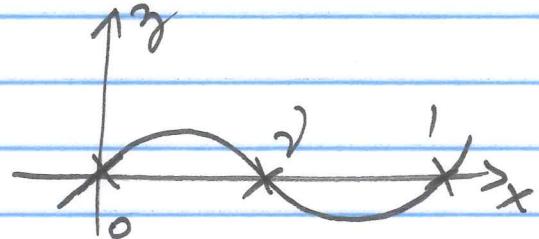
x	Exact u_1	Approx u_1	% error
0	0	0	0
$\frac{1}{4}$	1	1.0269	2.7
$\frac{1}{2}$	1.4142	1.3693	3.1

not as good as
for e' value

2) Exercise: approximate λ_2 and u_2 .

Introduce a polynomial trial function -
select one node (one zero) middle $(0, 1)$.

$$u(x) = A_2 x(\gamma-x)(1-x)$$



find A_2 and γ

(location of node) and λ_2
via above method

(look at DE and BCs)

Analyzing a problem, we can say that the node is in the middle. Otherwise, introduce v and find it.

Note $p(u)$ (and now with normalization,
 $\tilde{I}(u)$) is stationary at $u = e'$ function
and p is e' value. So, when establishing
the smallest e' value, we know that
 $p(u)$ has a min here and all
estimations of λ_1 , using $p(u)$, return
values greater than or equal to
the min:

$$\lambda_1 \leq p(u) \quad \forall u \in D_E$$

exact

and $p(u)$ provides an upper bound for λ_1 .