

12/08/2017

Ex Find solution of the BVP

$$\frac{d^2u}{dx^2} = f(x) \quad x \in (0, 1)$$

$$B_1 u = u(0) = C_1, \quad B_2 u = u(1) = C_2$$

as a series in terms of orthonormalized e'functions of the homogeneous problem.

One can write

$$u = u_f + u_c$$

$$u_f$$

$$u'' = f$$

$$u_c$$

$$u'' = 0$$

$$u(0) = u(1) = 0$$

$$u(0) = C_1$$

u_f as in
notes

Closed form
solution

$$u_c = C_1 + (C_2 - C_1)x$$

We will find an e'function expansion of $-u$ ($= u_f + u_c$) in one step.

Eigenvalue system: $Lu = -u'' = \lambda u, \quad x \in (0, 1)$

$$u(0) = u(1) = 0$$

We solved this problem previously

$$-u'' = -f$$

$$\lambda_n = (n\pi)^2$$

$n=1, 2, \dots$

$$u_n = \sqrt{2} \sin nx,$$

We will develop a Fourier sine series

Governing ODE:

u_n :

$$-u'' = -f$$

$$(*) \quad \int_0^1 u_n (-u'') dx = \int_0^1 u_n (-f) dx$$

and an eigenfunction
expansion of u is

$$u = \sum_{n=1}^{\infty} a_n u_n(x)$$

Orthogonality \Rightarrow $\langle u_n, u \rangle = a_n$
with inner product

$$\langle u, v \rangle = \int_0^1 u(x) v(x) dx$$

Lagrange identity / integration by parts twice
on (*). Then

$$\text{LHS} = - \int_0^1 u_n u'' dx \stackrel{\substack{\text{by} \\ \text{parts}}}{=} -[u_n u']_0^1 + \int_0^1 u_n' u' dx$$

$$\stackrel{\substack{\text{by} \\ \text{parts}}}{=} -[u_n u' - u u_n']_0^1 - \int_0^1 u_n'' u dx$$

Note sign
 $Lu = -u'' = -f$

$$\begin{aligned}
 -\int_0^1 u_n'' u dx &= \int_0^1 \underbrace{L u_n \cdot u}_{''} dx = 2n \int_0^1 u_n u dx = \\
 &\quad 2n u_n \\
 &= 2n \langle u_n, u \rangle = 2n a_n
 \end{aligned}$$

Boundary term

$$\begin{aligned}
 [\mathcal{J}(u_n, u)]' &= -[u_n u' - u u_n']' = [u u_n' - u_n u']' = \\
 &= u(1)u_n'(1) - u_n(1)u'(1) - u(0)u_n'(0) + u_n(0)u'(0) \\
 &\quad \left. \begin{array}{l} \text{since } u_n \text{ is } e^{f^2} \\ \text{is a sol} \end{array} \right) \quad \left. \begin{array}{l} \text{since } u_n \text{ is } e^{f^2} \\ u \text{ is bounded} \end{array} \right) \\
 &= C_2 u_n'(1) - C_1 u_n'(0)
 \end{aligned}$$

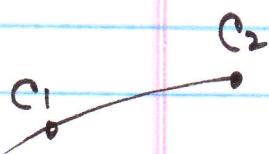
$$= C_2 u_n'(1) - C_1 u_n'(0)$$

Substitute the above in (*):

$$C_2 u_n'(1) - C_1 u_n'(0) + 2n a_n = - \int_0^1 u_n f dx$$

$$2n a_n = - \int_0^1 u_n f dx + C_1 u_n'(0) - C_2 u_n'(1)$$

$$q_n = \frac{-\int_0^1 u_n f dx + c_1 u_n'(0) - c_2 u_n'(1)}{2n}, \quad n=1, 2, \dots$$



~~* * * *~~ e'function
at endpoints

at endpoints
for inhomog. BCs
we have Gibb's
phenomenon

response to inhomogeneous BCs
makes convergence of $\sum_{n=1}^{\infty} q_n u_n$
for u slower

Response to inhomog. BCs:

$$q_n^c = \frac{c_1 u_n'(0) - c_2 u_n'(1)}{2n} = \frac{c_1 - c_2 (-1)^n}{(n\pi)^2} \sqrt{2n\pi}$$

from
 $u_n'(0)$
 $u_n'(1)$

$$2n = (n\pi)^2$$

$$u_n(x) = \sqrt{2} \sin n\pi x, \quad n=1, 2, \dots$$

$$u_n'(x) = \sqrt{2} n\pi \cos n\pi x$$

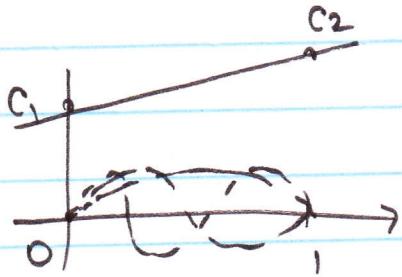
$$u_n'(0) = \sqrt{2} n\pi, \quad u_n'(1) = \sqrt{2} n\pi (-1)^n$$

$$\Rightarrow q_n^c = \sqrt{2} \frac{c_1 + (-1)^{n+1} c_2}{n\pi}$$

$$q_n^c = O\left(\frac{1}{n}\right) \text{ as } n \rightarrow \infty$$

$$u_n^c = \sum_{n=1}^{\infty} a_n^c u_n(x)$$

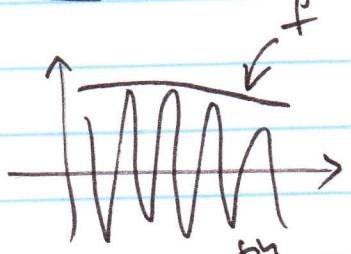
where the sum on the RHS converges to u_n^c in the mean square sense (in L^2), i.e. no absolute & uniform convergence.



Let

$$a_n^f = \frac{- \int_0^1 u_n f dx}{2\pi} = \frac{- \int_0^1 \sqrt{2\sin n\pi x} \cdot f(x) dx}{(n\pi)^2}$$

As $n \rightarrow \infty$ $\int_0^1 \sin n\pi x dx = o(1)$



i.e. numerator $\rightarrow 0$ as $n \rightarrow \infty$ (by Riemann-Lebesgue lemma)

So,

$$a_n^f = o\left(\frac{1}{n^2}\right)$$

so coefficients $a_n^f \rightarrow 0$ faster as $n \rightarrow \infty$ than coefficients a_n^c .

\int_L

$$u_f = \sum_{n=1}^{\infty} a_n f u_n(x)$$

the sum on the RHS converges to u_f absolutely and uniformly if f also satisfies homog. BCs $f(0) = f(1) = 0$



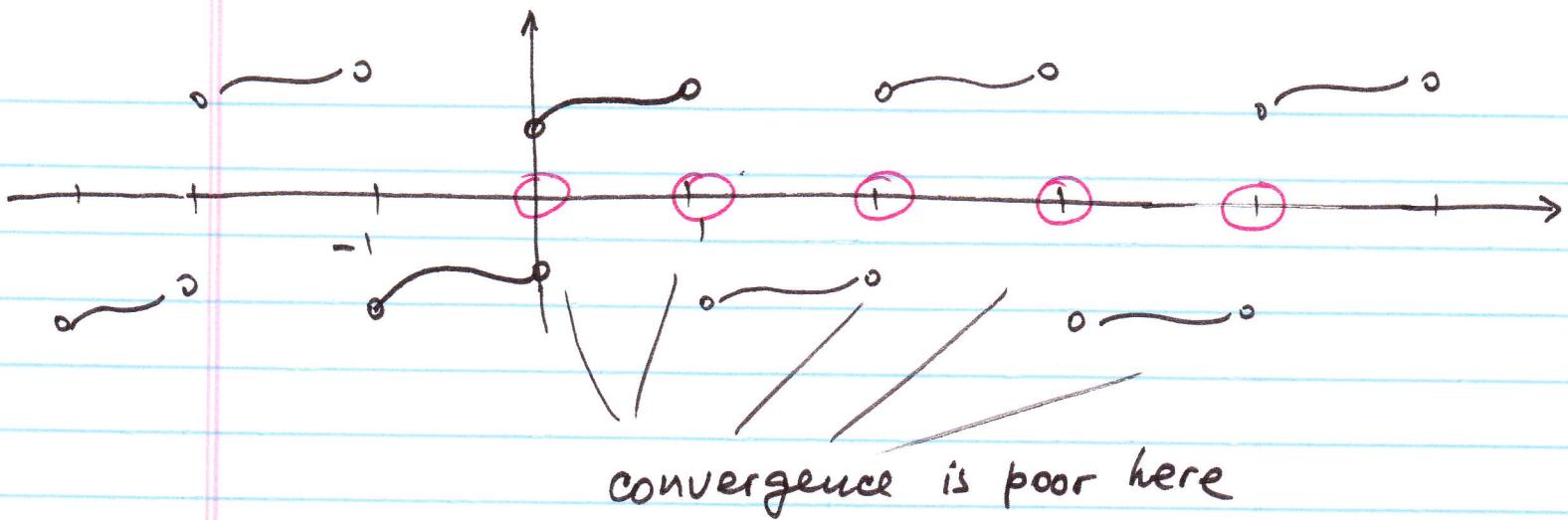
so, series for u_f has faster convergence (to u_f) than the series for u_c (convergence to u_c).

Note - function expansion is in terms of

$u_n = \sqrt{2} \sin nx$. The Fourier sine series will converge to λ -periodic odd extension.

For all $x \in (0, 1)$ $\sum_{n=1}^{\infty} a_n u_n$ gives the

odd-periodic extension of the solution of the BVP



$$\underline{\underline{Ex}} \quad (L - \mu) u = f(x) \quad x \in (0, 1)$$

$$B_1 u = u'(0) = C_1$$

$$B_2 u = u'(1) = C_2$$

Solve in terms of eigenfunctions of L.

$$Lu = -u''$$

$$\text{Eigenfunctions : } Lu = \lambda u \quad -u'' = \lambda u \\ u'(0) = 0 \quad u'(1) = 0$$

$$\text{Eigenvalues: } \lambda_0 = 0 \quad u_0 = 1$$

$$\lambda_n = (n\pi)^2 \quad u_n = \sqrt{2} \cos n\pi x, \quad n=1, 2, \dots$$

We could write

$$u = u_f + u_c$$

$$u_f \quad (L - \mu)u = f \quad u_c \quad (L - \mu)u = 0$$

$$u'(0) = u'(1) = 0 \quad u'(0) = c_1 \\ u'(1) = c_2$$

We have closed form solution for u_c .

Then we can develop an eigenfunction expansion for u in one go.

Form

$$\int_0^1 u_n (L - \mu) u dx = \int_0^1 u_n f dx$$

where the solution u has expansion

$$u = \sum_{n=0}^{\infty} a_n u_n(x) \text{ so that } \langle u_n, u \rangle = a_n$$

$$\text{Here again } \langle u, v \rangle = \int_0^1 uv dx$$

$$\int_0^1 u_n (-u'' - \mu u) dx = \int_0^1 u_n f dx$$

apply
Lagrange identity

In the LHS

$$-\int_0^1 u_n u'' dx = \begin{matrix} \text{twice} \\ \text{by parts} \end{matrix} - [u_n u' - u'_n u]_0^1 - \int_0^1 u_n''' u dx$$

$$[u_n u' - u'_n u]_0^1$$

now

$$-\int_0^1 u_n'' u dx = 2\lambda_n \int_0^1 u_n u dx = 2\lambda_n a_n$$

and

$$[u_n u'_n - u_n' u'_n]_0^1 = u(1) u'_n(1) - u_n(1) u'(1) -$$

$$- u(0) u'_n(0) + u_n(0) u'(0)$$

$$\begin{matrix} & & \\ & & \\ =0 & & c_1 \\ & & \end{matrix}$$

u_n is $e^{f(x)}$

do,

$$(*) \quad c_1 u_n(0) - c_2 u_n(1) + (\lambda_n - \mu) a_n = \int_0^1 u_n f dx$$

for $\mu \neq \lambda_n$ $\forall n=1, 2, \dots$, i.e. μ is not an eigenvalue,
we have a_n

$$a_n = - \frac{\int_0^1 u_n f dx + c_2 u_n(1) - c_1 u_n(0)}{\mu - \lambda_n}$$

Note For large n , i.e. in the limit $n \rightarrow \infty$

$$\int_0^1 u_n f dx = o(1): \quad c_2 u_n(1) - c_1 u_n(0) =$$

as $n \rightarrow \infty$

$$= (-c_1 + (-1)^n c_2) \sqrt{2} = O(1)$$

as $n \rightarrow \infty$

a_n and hence $\mu - \lambda_n$ are $O(n^2)$ as $n \rightarrow \infty$

$$a_n^f = O\left(\frac{1}{n^2}\right) \quad a_n^c = O\left(\frac{1}{n^2}\right)$$

again the convergence of $\sum a_n^c u_n$ to u_c is slower than the convergence of $\sum a_n^f u_n$ to u_f . Series expansion $\sum a_n u_n$

is the even 2-periodic extension of the solution u of the BVP and is its Fourier cosine series.

E' function expansion of G for BVP in this case

$$G(x, \xi) = - \sum_{n=0}^{\infty} \frac{u_n(x) u_n(\xi)}{\mu - \lambda_n}$$

If $\mu = \lambda_m$ for some m , then μ is an eigenvalue. (*) holds for $\neq n \neq m$, but for $n = m$, there is a bounded a_m if and only if

$$(A) \quad \int_0^1 u_m f dx - c_1 u_m(0) + c_2 u_m(1) = 0$$

This is, of course, exactly the solvability condition on the data (c_1, c_2, f) that is necessary and sufficient for a solution u to the BVP to exist — as given also by the Fredholm Alternative.

Now, given (A) there is a solution, and it is not unique

$$u = \sum_{n=0}^{\infty} a_n u_n(x) + c u_m(x)$$

|
arb. const

and we have the e^f expansion of $G_M(x, \xi)$:

$$G_M(x, \xi) = - \sum_{n=0}^{\infty} \frac{u_n(x) u_n(\xi)}{\lambda_m - \lambda_n}$$

|
 $n \neq m$

$$u_f = \sum_{n=0}^{\infty} a_n^f u_n$$

$$a_n^f = \frac{- \int_0^1 u_n f dx}{\mu - \lambda_n}$$

$$u_c = \sum_{n=0}^{\infty} a_n^c u_n$$

|
 $n \neq m$

$$a_n^c = \frac{c_1 u_n(0) - c_2 u_n(1)}{\mu - \lambda_n}$$

$$\mu \neq \lambda_n$$