

Ex 2 (Cont'd)

$$L = e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx} + 1 = \frac{d}{dx} (e^x \frac{d}{dx}) + 1$$

$x \in (0, 1)$

$$B_1 u = u'(0)$$

$$B_2 u = u(0) + u(1)$$

We know that  $L = L^*$ , i.e.  $L$  is formally self-adjoint. Last time we found

$$J(u, v)'_0 = [e^x (u'v - uv')]_0'$$

To find  $B_i^*$ , we set  $J(u, v)'_0 = 0$ .

$$0 = e^x (\underbrace{u'(0)v(1) - u(0)v'(1)}_{\sim} - \underbrace{(u'(0)v(0) - u(0)v'(0))}_{\sim} - u(1)v'(0))$$

$$u \in D_B \Rightarrow u'(0) = 0 \quad B_1 u = 0$$

$$u(0) + u(1) = 0 \quad B_2 u = 0$$

$$= e^x u'(1)v(1) + u(1) (e^x v'(1) - v'(0))$$

Since  $u'(1)$  and  $u(1)$  are arbitrary, we get

$$\begin{aligned} ev'(1) &= 0 && \text{homogeneous} \\ -ev'(1) - v'(0) &= 0 && \text{adjoint boundary} \\ &&& \text{conditions} \end{aligned}$$

or

$$\begin{aligned} B_1^*v = v(1) &= 0 && (\text{We can take any linear combination}) \\ B_2^*v = v'(0) + ev'(1) &= 0 && \text{of these BCs} \end{aligned}$$

Notice  $B_1 \neq B_1^*$   $B_2 \neq B_2^*$

i.e. BCs and adjoint BCs, or more precisely, boundary operators  $B_i$  and adjoint boundary operators  $B_i^*$  are not the same.

So, problem is not self-adjoint (it is only formally self-adjoint).

$$D_{B^*} = \{v : v \in C^2(0,1) \text{ and } B_i^*v = 0\}$$

$$\mathcal{L}^* = \{L^*, D_{B^*}\} = \{L, D_{B^*}\}$$

$$\mathcal{L} = \{ L, D_B \} \neq \mathcal{L}^* \quad \text{since } D_B^* \neq D_B$$

Ex 3

$$L = \frac{d}{dx} \left( a_0(x) \frac{d}{dx} \cdot \right) + a_2(x) \quad x \in (a, b)$$

We see that  $L$  is formally self-adjoint.

In the Frobenius method, if  $a_0(x) = 0$ , we  
(power series method)

can have a singular point      method for solving  
ODEs

assume

!  $(a_0(x) \neq 0, x \in (a, b))$ , to ensure solutions  
of  $Lu=0$  are bounded on  $[a, b]$ , i.e.  
BVP is regular, not singular.

$$\begin{array}{ll} c_0, c_1, & B_1 u = c_0 u(a) + c_1 u'(a) \\ b_0, b_1 : & \text{separated BCs} \\ \text{arbitrary} & B_2 u = b_0 u(b) + b_1 u'(b) \\ \text{constants} & (\text{only at one endpoint}) \end{array}$$

These boundary conditions are the general separated/unmixed BCs for a 2<sup>nd</sup> order problem, meaning that each BC involves  $u$  and  $u'$  at just one of the endpoints of interval,  $x=a, x=b$  (order is not important).

Ex.

$$u(0)=0 \quad \text{separated BC}$$

$$u(0)+u(1)=0 \quad \text{not separated BC / mixed}$$

$$\langle fg \rangle = \int_a^b f g \, dx$$

Consider

$$\langle v, Lu \rangle = \int_a^b v \left( (a_0 u')' + a_2 u \right) \, dx \quad \begin{array}{l} \text{by} \\ \text{parts} \end{array}$$

$v \in C^2(a, b)$  arbitrary

$u \in D_B$  arbitrary

$$D_B = \{u : u \in C^2(a, b), \quad B_i u = 0\}$$

$$\begin{aligned} &= [a_0 (u'v - uv')]_a^b + \int_a^b u \underbrace{\left( (a_0 v')' + a_2 v \right)}_{L^* v} \, dx \\ &\quad \left. J(u, v) \right|_a^b \\ &\quad \underbrace{\qquad\qquad\qquad}_{\langle L^* v, u \rangle} \end{aligned}$$

$L = L^*$   $\Rightarrow L$  is formally self-adjoint  
(at least)

Construct  $B_i^*$ .

$$[J(u, v)]_a^b = 0 \quad -\frac{b_1}{b_0} u'(b)$$

$$0 = a_0(b) [u'(b)v(b) - u(b)v'(b)]$$

$$-a_0(a) [u'(a)v(a) - u(a)v'(a)] \Leftrightarrow -\frac{c_1}{c_0} u'(a)$$

If  $c_0 \neq 0, b_0 \neq 0$ , then  $B_i u = 0$  give

$$u(a) = -\frac{c_1}{c_0} u'(a) \quad B_1 u = 0$$

$$u(b) = -\frac{b_1}{b_0} u'(b) \quad B_2 u = 0$$

$$\Leftrightarrow a_0(b) u'(b) [v(b) + \frac{b_1}{b_0} v'(b)]$$

$$-a_0(a) u'(a) [v(a) + \frac{c_1}{c_0} v'(a)]$$

Since  $u'(a), u'(b)$  are arbitrary,  $\dots = 0, \dots$

$$B_1^* v = c_0 v(a) + c_1 v'(a) = 0 \quad (\text{order of } B_i^*)$$

$$B_2^* v = b_0 v(b) + b_1 v'(b) = 0 \quad \text{does not matter}$$

Notice,  $B_1^* = B_1$ ,  $B_2^* = B_2$ , i.e.  $B_i^* = B_i$ . and  
so  $D_B = D_{B^*}$ .

$$\text{So, } \mathcal{L}^* = \{L^*, D_{B^*}\} = \{L, D_B\} = \mathcal{L}$$

adjoint  
boundary value  
diff. operator

so the BVP operator  $\mathcal{L}$  is self-adjoint.

Def An operator  $\mathcal{L} = \{L, D_B\}$  is self-adjoint  
if it is equal to its adjoint  
 $\mathcal{L}^* = \{L^*, D_{B^*}\}$ .

Note if one or both  $c_0$  and  $b_0$  are zero,  
the result  $\mathcal{L}^* = \mathcal{L}$  still holds.

Analogy between BVPs and linear  
algebra

Real vector space,  $\vec{u} \in \mathbb{R}^n$ ,  $n \times n$  matrix  
A is self-adjoint if it is equal

to its adjoint  $A^T$ , i.e.  $A = A^T$ , so  $A$  is symmetric.

$\Rightarrow \lambda$ 'values are real, and e'vectors span  $\mathbb{R}^n$ , they form a basis of  $\mathbb{R}^n$ .

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$$

inner product dot product

$$\langle \vec{v}, A\vec{u} \rangle = \langle A\vec{v}, \vec{u} \rangle : \text{result of } A \text{ being self-adjoint}$$

Complex space  $\vec{u} \in \mathbb{C}^n$ ,  $n \times n$ -matrix  $A$  is Hermitian if

$$\underline{A = \bar{A}^T}$$

|  
analogue of self-adjoint in complex case

$$\langle \vec{u}, \vec{v} \rangle \stackrel{\text{def}}{=} \vec{u} \cdot \overline{\vec{v}}$$

'conjugate'

The structure of  $\lambda$ 'values and e'vectors make them useful in theory and computations.