

9/1/2017

Delta - sequences

$S_n(x)$: classical functions

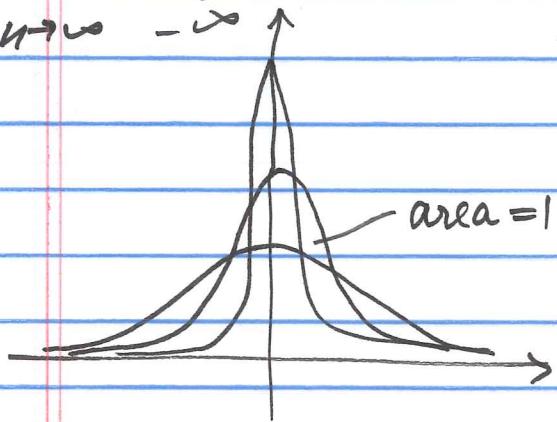
$$\lim_{n \rightarrow \infty} S_n(x) = \delta(x)$$

$$\lim_{n \rightarrow \infty} S_n(x) = 0 \quad x \neq 0$$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} S_n(x-\xi) \phi(\xi) d\xi = \phi(x)$$

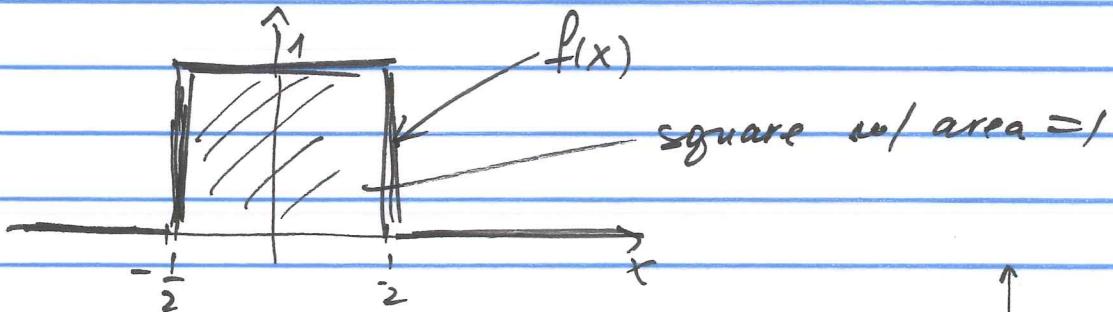
test function

Ex

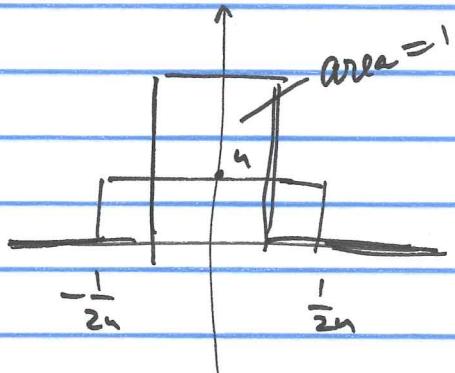


as $n \uparrow$, functions become more localized with bigger values around 0

Ex



$$n f(nx) = \begin{cases} n, & |x| < \frac{1}{2n} \\ 0, & |x| > \frac{1}{2n} \end{cases}$$



$$G: L G(x, \xi) = \delta(x - \xi)$$

Instead of solving this equation, we solve

$$L G_n = S_n(x - \xi) \text{ with classical functions.}$$

$$L G_n \rightarrow L G, \quad S_n \rightarrow \delta.$$

We need to prove that the resulting G doesn't depend on a particular choice of S_n .

Def Derivative of function $f: f'$

$$(i) \quad \langle f', \phi \rangle = - \langle f, \phi' \rangle \quad \forall \phi \in C_c^\infty$$

↑
test function

ϕ are smooth enough
and have compact
support

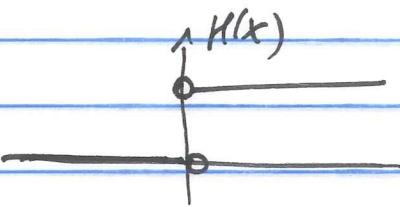
Def Derivative of order n : $f^{(n)}$

$$(ii) \quad \langle f^{(n)}, \phi \rangle = (-1)^n \langle f, \phi^{(n)} \rangle \quad \forall \phi \in C_c^\infty$$

$H(x)$ Heaviside step function is a

distribution

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$



$$H'(x) = \delta(x)$$

We treat $\frac{dH}{dx} = \delta(x)$ in the distribution sense.

This equation only makes sense when applied to functions. This follows from def (i).

$$\left\langle \frac{dH}{dx}, \phi \right\rangle \stackrel{(i)}{=} - \langle H, \phi' \rangle \quad \forall \phi \in C_0^\infty$$

$$= - \int_{-\infty}^{\infty} H(x) \phi'(x) dx \quad \textcircled{=} \quad \text{---}$$

$$\langle \cdot, \cdot \rangle = \int_{-\infty}^{\infty} \cdot \cdot \cdot dx$$

$$\textcircled{=} - \int_0^{\infty} \phi'(x) dx = - (\phi(\infty) - \phi(0)) = \phi(0)$$

ϕ has compact support

□

$$\int_{-\infty}^{\infty} \delta(x-0) \phi(x) dx = \phi(0)$$

"

$$\langle \delta(x), \phi \rangle$$

$\exists \langle \delta(x), \phi \rangle$

Hence, $\frac{dH}{dx} = \delta(x).$

3. Green's function methods for ODEs

3.1 Statement of the problem

The aim is to solve BVP (f.2)

$$Lu = f(x), \quad x \in (a, b)$$

(3.1)

$$B_i u = c_i \quad i=1, \dots, n$$

Recall

$$L = \sum_{j=0}^n a_j(x) \frac{d^{n-j}}{dx^{n-j}} \quad : n^{\text{th}} \text{ order linear diff. operator}$$

$a_j(x)$: crs on (a, b)

$a_0(x) \neq 0, \quad x \in (a, b)$

$$B_i: u = \sum_{j=1}^n \alpha_{ij} \frac{d^j}{dx^j} u(a) + \beta_{ij} \frac{d^j}{dx^j} u(b)$$

mixed BCs

matrix $(\alpha | \beta)$ has rank n

BVP may have a unique solution, no solution or ∞ many solutions.

$$\underline{\text{Ex}} \quad Lu = u'' + \pi^2 u = 0 \quad x \in (0, 1)$$

$$B_1: u = u(0) = 0, \quad B_2: u = u(1) = 1$$

$$(1) \quad u'' + \pi^2 u = 0 \quad r^2 + \pi^2 = 0 \quad r$$

Assume $u = e^{rx}$,

$$u' = re^{rx}, \quad u'' = r^2 e^{rx}$$

$$r^2 e^{rx} + \pi^2 e^{rx} = 0 \Rightarrow e^{rx} (r^2 + \pi^2) = 0$$

$\neq 0$

$\Rightarrow r^2 + \pi^2 = 0$: characteristic equation

$$r = \pm i\pi \Rightarrow e^{\pm i\pi x} = \underbrace{\cos \pi x}_{\text{also solution of (1)}} + i \underbrace{\sin \pi x}_{\text{solution of (1)}}$$

also solution of (1)

$$u(x) = C_1 \overset{x=0}{\cancel{\cos}} ux + C_2 \overset{x=0}{\cancel{\sin}} ux$$

$$u(0) = 0 \Rightarrow u(0) = C_1 \overset{x=0}{\cancel{\cos}} 0 + C_2 \overset{x=0}{\cancel{\sin}} 0 \Rightarrow C_1 = 0$$

$$u(x) = C_2 \overset{x=0}{\cancel{\sin}} ux$$

$$u(1) = 1 \Rightarrow u(1) = C_2 \overset{x=1}{\cancel{\sin}} \pi \Rightarrow 1 = 0 \quad \text{↯}$$

\therefore BVP has no solutions

Edwards & Penney ODEs book

Regard functions $a_{ij}(x)$, α_{ij} , β_{ij} given and fixed, so that L and B_i are given and

fixed. Then consider the inhomogeneous terms $\{f(x), c_i\}$, which are referred to as the "data" for the BVP. Question is how does the solution $u(x)$ depend on the data?

The analog in linear algebra:

$$Ax = b$$

matrix | vector, data
solution

A, b are given, find x .

There are two ways.

1st: use Gaussian elimination. A is fixed, If b is changing, then we would need to recompute Gaussian elimination for every b .

2nd: Consider A to be fixed

$$x = A^{-1}b$$

so, if we find inverse matrix A^{-1} , then we have a solution. But finding A^{-1} may be difficult.

With BVPs, we use the second approach.

Def The Green's function $G(x, \xi)$ of (3.1) is the solution of

$$L G(x, \xi) = \delta(x - \xi) \quad x, \xi \in (a, b)$$

$$B_i G = 0 \quad i=1, \dots, n$$

Note

(1) Equation $L G = \delta$ has to be interpreted in the sense of distribution. Equivalently,

we can replace $LG = \delta$ by

$$(i) \quad x \neq \xi \quad LG = 0$$

$$(ii) \quad x = \xi \quad \frac{d^k G}{dx^k} \text{ are continuous for } k = 0, 1, 2, \dots, n-2$$

$\frac{d^{n-1} G}{dx^{n-1}}$ is discontinuous at $x = \xi$

with jump

$$\left[\frac{d G^{n-1}}{dx^{n-1}} \right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{a_0(\xi)} \quad (3.2')$$

These conditions hold in the classical sense and they agree w/ def of delta function.

$$x \neq \xi \Rightarrow \delta(x - \xi) = 0, \text{ so is } LG = 0$$

$$G \in C^n(a, b) \text{ for } x \neq \xi$$

$$x = \xi: a_0(x) \frac{d^n f}{dx^n} + a_1(x) \frac{d^{n-1} f}{dx^{n-1}} + \dots + a_n(x) f = \delta(x - \xi)$$