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$$x = \xi: \quad a_0(x) \frac{d^n G}{dx^n} + a_1(x) \frac{d^{n-1} G}{dx^{n-1}} + \dots + a_n(x) G = \delta(x - \xi)$$

Integrate across  $x = \xi$ :  $\int_{\xi^-}^{\xi^+} \dots dx$ .

On the right  $\int_{\xi^-}^{\xi^+} \delta(x - \xi) dx = 1$

Recall that  $G$  has CTB derivatives up to and including order  $n-2$ , and  $(n-1)^{th}$  order derivatives has a finite jump, like Heaviside step function  $H$ , so that  $n^{th}$  order derivative behaves like  $\delta$ . Then on the LHS

$$\int_{\xi^-}^{\xi^+} \left( \underbrace{a_0(x)}_{\text{cont}} \frac{d^n G}{dx^n} + \underbrace{a_1(x)}_{\text{cont}} \frac{d^{n-1} G}{dx^{n-1}} + \dots + \underbrace{a_n(x)}_{\text{cont}} G \right) dx =$$

$\underbrace{\hspace{15em}}_{\text{cont (after } \int)}$

no contribution

$$= a_0(\xi) \left[ \frac{d^{n-1} G}{dx^{n-1}} \right]_{\xi^-}^{\xi^+}$$

Hence,

$$\rho_0\left(\frac{\rho}{\xi}\right) \left[ \frac{d^{n-1} G}{dx^{n-1}} \right]_{\xi^-}^{\xi^+} = 1$$

Note (2)

The BCs for  $G$  are homogeneous.

If we extend the definition of  $D_B$ , domain of  $L$ , by not requiring  $u \in C^n(a, b)$ , then  $G \in D_B$ , i.e. the Green's function is in domain of  $L$ .

Note (3) The solution of (3.2) for  $G$  is unique if it exists, i.e. if there is a solution of (3.2), then it is unique. It exists if the completely homogeneous problem  $\{f, c_i\} = \{0, 0\}$

$$Ly = 0$$

$$B_i y = 0 \quad i = 1, 2, \dots, n$$

has no nontrivial solution, i.e. when

the homogeneous problem has only trivial solution.

BVP

$$Lu = f$$

$$B_i u = c_i \quad i=1, \dots, n$$

Green's function

$$LG = \delta(x - \xi)$$

$$B_i G = 0$$

Test to see if there is  $G$ :

$$Ly = 0$$

$$B_i y = 0$$

only solution is  $y \equiv 0 \Leftrightarrow G$  exists

Note (4) Physically,  $G(x, \xi)$  is the response at  $x$  when the only forcing is a point source  $\delta(x - \xi)$  concentrated at  $x = \xi$ .

### 3.2 Construction of $G(x, \xi)$ , $n=2$ separated/unmixed BCs

Construction for general  $n$  is the same in principle. In this case,  $G(x, \xi)$  satisfies

$$Lu \equiv a_0(x)u'' + a_1(x)u' + a_0(x)u = \delta(x - \xi)$$

$$(3.3) \quad B_1 u = \alpha_1 u(a) + \beta_1 u'(a) = 0$$

$$B_2 u = \alpha_2 u(b) + \beta_2 u'(b) = 0$$

- 1) Find two (in general case  $n$ ) linearly independent solutions of  $Lu=0$ . Call them  $\tilde{u}_1$  and  $\tilde{u}_2$ . Now, find those linear combinations of  $\tilde{u}_1, \tilde{u}_2$  which satisfy one BC (and not the other). Call these linear combinations  $u_1$  and  $u_2$ , i.e.

$$u_1 = \mu_1 \tilde{u}_1 + \nu_1 \tilde{u}_2$$

Find  $\mu_1, \nu_1$  such that  $B_1 u_1 = 0$ .

$$B_1 u_1 = \alpha_1 (\mu_1 \tilde{u}_1(a) + \nu_1 \tilde{u}_2(a)) + \beta_1 (\mu_1 \tilde{u}_1'(a) + \nu_1 \tilde{u}_2'(a)) =$$

$$= \mu_1 \left( \alpha_1 \tilde{u}_1(a) + \beta_1 \tilde{u}_1'(a) \right) + \nu_1 \left( \alpha_1 \tilde{u}_2(a) + \beta_1 \tilde{u}_2'(a) \right) = 0$$

determines the ratio  $\mu_1$  to  $\nu_1$ , and hence determines  $u_1$  up to a multiplicative constant.

Then

$$u_2 = \mu_2 \tilde{u}_1 + \nu_2 \tilde{u}_2$$

is to satisfy  $B_2 u_2 = 0$

$$B_2 u_2 = \mu_2 B_2 \tilde{u}_1 + \nu_2 B_2 \tilde{u}_2$$

gives  $u_2$  up to a multiplicative constant.

Then  $G$  is proportional to  $u_1$  and  $u_2$  to left and right of  $x = \xi$ . Put

$$G(x, \xi) = \begin{cases} A u_1(x) u_2(\xi), & a < x < \xi < b \\ A u_1(\xi) u_2(x), & a < \xi < x < b \end{cases}$$

then automatically, the choice of multiple of  $u_1$  and  $u_2$  to left and right of  $x = \xi$  is such that  $G(x, \xi)$  is continuous at  $x = \xi$ .