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2) Apply the jump condition at $x = \xi$:

$$a_0(\xi) \left[\frac{\partial G}{\partial x} \right]_{x=\xi^-}^{x=\xi^+} = 1$$

This determines constant A .

$$\frac{\partial G(x, \xi)}{\partial x} = \begin{cases} A u_1'(x) u_2(\xi) & a \leq x < \xi \leq b \\ A u_1(\xi) u_2'(x), & a \leq \xi < x \leq b \end{cases}$$

$$a_0(\xi) \left[A u_1(\xi) u_2'(\xi) - A u_1'(\xi) u_2(\xi) \right] = 1$$

$$a_0(\xi) A \left[u_1(\xi) u_2'(\xi) - u_1'(\xi) u_2(\xi) \right] = 1$$

Notice that

$$u_1(\xi) u_2'(\xi) - u_1'(\xi) u_2(\xi) = \begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1'(\xi) & u_2'(\xi) \end{vmatrix}$$

$= W(u_1, u_2)(\xi)$ is the Wronskian of

u_1 and u_2 .

Hence,

$$a_0(\xi) A \cdot W(u_1, u_2)(\xi) = 1$$

$$\therefore \boxed{A = \frac{1}{a_0(\xi) W(u_1, u_2)(\xi)}}$$

(BVP has a unique solution ensures that $\bar{W} \neq 0$)

If $W(u_1, u_2)(\xi) \neq 0 \quad \forall \xi \in [a, b]$, then we can construct G , ϕ .

$W(u_1, u_2)(\xi) \neq 0$ is related to the existence of $G(x, \xi)$.

We have constructed

$$(3.4) \quad f(x, \xi) = \begin{cases} \frac{u_1(x) u_2(\xi)}{a_0(\xi) W(u_1, u_2)(\xi)}, & a \leq x < \xi \leq b \\ \frac{u_1(\xi) u_2(x)}{a_0(\xi) W(u_1, u_2)(\xi)}, & a \leq \xi < x \leq b \end{cases}$$

$$Ly=0, B_i y=0$$

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Lemma The homogeneous problem only

has the trivial (zero) solution $\Leftrightarrow \overset{(ii)}{W}(u_1, u_2) \neq 0$
for $x \in [a, b]$ $\Leftrightarrow G(x, \xi)$ exists.

(i)

Proof (i) By construction of (3.4) for
 $G(x, \xi)$, G exists $\Leftrightarrow \overset{(i)}{W} \neq 0$.

(ii) Show this by logical contraposition:
the homogeneous problem has a non-trivial
solution $\Leftrightarrow \overset{(ii)}{W}(u_1, u_2) = 0$ at some point
 $x_0 \in [a, b]$.

\Rightarrow Assume that $\exists y \neq 0: Ly=0, B_i y=0$.

Then choose $u_1(x) = y, u_2(x) = y$. Then

u_1 and u_2 are linearly dependent and

"Wronskian is zero"

nontrivial
linear
combination of
 u_1, u_2

$$c_1 u_1(x) + c_2 u_2(x) = 0$$

$$c_1^2 + c_2^2 \neq 0$$

at least one
const is $\neq 0$

$$c_1 u_1'(x) + c_2 u_2'(x) = 0$$

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix}$ is

singular $\Rightarrow W = \det(\dots) = 0$ on $[a, b]$

\Leftarrow (Will give a handout on Wronskian
- see Abel's formula)

If u_1 and u_2 are any solutions of
2nd order, linear, homogeneous ODE
and $W(u_1, u_2) = 0$ at some point \Rightarrow
by Abel's formula, $W(u_1, u_2) = 0$ at
 $\forall x \in [a, b]$. Hence u_1 and u_2 are
linearly dependent: $u_1 = \alpha u_2$.

Now, with u_1 , such that $B_1 u_1 = 0$
and $B_2 u_2 = 0$, we have

$$Lu_1 = 0, \quad Lu_2 = 0 \quad \text{but} \quad u_1 = d u_2$$

$$\Rightarrow B_1 u_2 = B_1 \cdot \frac{1}{2} u_1 = \frac{1}{2} B_1 u_1 = 0$$

$$\Rightarrow B_1 u_2 = 0$$

$$B_2 u_1 = B_2 d u_2 = d B_2 u_2 = 0$$

$$\Rightarrow B_2 u_1 = 0$$

Hence, any multiple of u_1 is a non-zero solution of the homogeneous BVP.

Note Result holds for any n .

Note Show for the above example w/ $n=2$ and separated BCs that

$$LG(x, \xi) = \delta(x - \xi)$$

in the sense of distributions, and $G(x, \xi)$

is as in (3.4).

We need to show

$$\langle LG(x, \xi), \phi \rangle = \langle \delta(x - \xi), \phi \rangle$$

for all test functions $\phi \in C_0^\infty$

$$\langle \cdot, \cdot \rangle = \int_a^b \cdot \cdot \cdot dx$$

∞ many times
differentiable and
w/ compact support

Def of $\delta(x - \xi)$ gives

RHS

$$\langle \delta(x - \xi), \phi \rangle = \int_a^b \delta(x - \xi) \phi(x) dx = \phi(\xi)$$

LHS Recall def of distributional derivatives
applied to LG :

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle \quad + \phi \in C_0^\infty$$

$$\langle f^{(n)}, \phi \rangle = (-)^n \langle f, \phi^{(n)} \rangle$$

which gives

$$\langle LG, \phi \rangle = \langle G, L^* \phi \rangle$$

$$Lu = a_0 u'' + a_1 u' + a_2 u$$

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where $L^* \phi = (a_0 \phi)'' - (a_1 \phi)' + a_2 \phi$
 is the formal adjoint operator acting
 on ϕ . So, we want to show

$$a_0 \cdot w / \quad \phi(\xi) = ? \langle G, L^* \phi \rangle \quad \forall \phi \in C_0^\infty$$

and we have (3.4) for G :

$$G(x, \xi) = \begin{cases} \frac{u_1(x) u_2(\xi)}{a_0(\xi) W(u_1, u_2)(\xi)}, & a \leq x \leq \xi \leq b \\ \frac{u_1(\xi) u_2(x)}{a_0(\xi) W(u_1, u_2)(\xi)}, & a \leq \xi \leq x \leq b \end{cases}$$

equivalently, we need to show

$$a_0(\xi) \bar{W}(\xi) \phi(\xi) = u_2(\xi) \int_a^{\xi} u_1(x) L^* \phi dx +$$

$$+ u_1(\xi) \int_{\xi}^b u_2(x) L^* \phi dx$$