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We need to show

$$a_0(\xi) W(\xi) \phi(\xi) = u_2(\xi) \int_a^c u_1(x) L^* \phi dx + \\ + u_1(\xi) \int_a^c u_2(x) L^* \phi dx$$

Integrate RHS.

$u_1, u_2 \in C^2(a, b)$, $\phi \in C_c^\infty(a, b)$. we can

use Lagrange's identity (integrate by parts and write in a specific form)

$$\int_a^c u_1(x) L^* \phi dx = \langle u_1, L^* \phi \rangle_{(a, \xi)} = \\ = \underbrace{\langle L u_1, \phi \rangle_{(a, \xi)}}_{u_1 \text{ is a solution}} + \underbrace{[J(u_1, \phi)]_a^c}_{\text{contribution from } x=a \text{ is zero since } Bu_1=0 \text{ (at } x=a)}$$

u_1 is a solution
of $L u_1 = 0$

contribution from $x=a$
is zero since $Bu_1=0$ (at $x=a$)

since $L u_2 = 0$

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$$\int_{\xi}^b u_2(x) L^* \phi dx = \langle u_2, L^* \phi \rangle = \langle L u_2, \overset{\circ}{\phi} \rangle_{(\xi, b)} + \\ + \left[J(u_2, \phi) \right]_{\xi}^b$$

contribution at $x=b$
is zero : $\therefore B_2 u_2 = 0$

so,

$$RHS = u_2(\xi) J(u_1, \phi) \Big|_{\xi} - u_1(\xi) J(u_2, \phi) \Big|_{\xi} \equiv$$

For this example,

$$J(u, v) = a_0 (v u' - u v') + (q_1 - a_0') u v$$

some

$$\equiv a_0(\xi) W(u_1, u_2)(\xi) \phi(\xi) = LHS$$

algebra

2) UNSEPARATED / MIXED BOUNDARY CONDITIONS

Periodic BCs are an example of mixed BCs

$$u'' + \lambda^2 u = 0 \quad x \in (0,1)$$

$$B_1 u = u(1) - u(0) = 0$$



$$B_2 u = u'(1) - u'(0) = 0$$

There are other examples as well.

u_1, u_2 approach no longer works. Take any two linearly independent solutions of homogeneous ODE \tilde{u}_1, \tilde{u}_2 . Hence,

$$\tilde{u}_1 = \sin 2x, \quad \tilde{u}_2 = \cos 2x$$

Put

$$G(x, \xi) = \begin{cases} A\tilde{u}_1(x) + B\tilde{u}_2(x), & 0 \leq x < \xi \leq 1 \\ C\tilde{u}_1(x) + D\tilde{u}_2(x), & 0 \leq \xi < x \leq 1 \end{cases}$$

A, B, C, D are constants but they depend on ξ

Now: 2 BCs, G is continuous at $x = \xi$ and

$$\left[\frac{\partial G}{\partial x} \right] \Big|_{x=\xi} = \frac{1}{g_0(\xi)}$$

Hence, we have 4 conditions \Rightarrow 4 linear eq^{"s} in 4 unknowns A, B, C, D .

3.3 Applications of Green's function

How to find $u(x)$ from $G(x, \xi)$ and the data $\{f, c_i\}$.

The basic result is (we discussed this previously):

Thm The Green's function $G(x, \xi)$ of (3.2)

$$(3.2) \quad L G(x, \xi) = \delta(x - \xi)$$

$$B_i G = 0 \quad i=1, \dots, n$$

is such that

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi \quad (3.9)$$

satisfies the BVP with homogeneous BCs, i.e. u satisfies

$$Lu = f \quad \text{and} \quad B_i u = 0, \quad i=1, \dots, n,$$

the BVP with data $\{f, \vec{\phi}\}$

Note

- 1) We are assuming that G exists so that (3.9) is defined, i.e. the homogeneous BVP only has the zero solution.
- 2) Result holds for general n and general BCs B_i (well-posed), i.e. works for mixed BCs too.

Proof Direct verification: check $Lu=f$ and $B_i u=0$. Seems OK naively:

$LG=\delta(x-\xi)$ and $B_i G=0$, but for

$L \int_a^b G(x, \xi) f(\xi) d\xi$. G is not smooth

enough to take $L \int = \int L$, i.e. to swap L and \int and then integrate over $x=\xi$.

Specific example: G of (3.3) and (3.4), $n=2$, BCs are separated / unmixed.

Then in (3.9),

$$u(x) = u_2(x) \int_a^x \frac{u_1(\xi) f(\xi)}{q_0(\xi) W(\xi)} d\xi + u_1(x) \int_x^b \frac{u_2(\xi) f(\xi)}{q_0(\xi) W(\xi)} d\xi$$

Now we can take derivatives wrt x .

Ex $Lu = u'' + 2^2 u = f(x)$ $x \in (0,1)$

$\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$: given parameter

(3.10)

$$B_1 u = u(0) = 0$$

$$\langle \cdot, \cdot \rangle = \int_0^1 \cdot \cdot \cdot dx$$

$$B_2 u = u(1) + u'(1) = 0$$

$G(x, \xi)$ satisfies $LG = \delta(x - \xi)$ and $B_i G = 0$

$Lu = 0$ has linearly independent solutions
 $\sin \lambda x$ and $\cos \lambda x$

Find linear combinations u_1 and u_2
s.t. $B_1 u_1 = 0$ and $B_2 u_2 = 0$

$u_1 = \sin \lambda x$

$$\begin{aligned} u_1(a-b) &= \sin \lambda a \cos \lambda b \\ &\quad - \cos \lambda a \sin \lambda b \end{aligned}$$

u_2 : put

$u_2 = a \sin 2(x-1) + b \cos 2(x-1)$: for convenience
to satisfy BC
at $x=1$

$$u_2(1) + u_2'(1) = 0$$

$$\Rightarrow b + a2 = 0$$

let $a=1 \Rightarrow b=-2$. So,

$$u_2(x) = \sin 2(x-1) - 2 \cos 2(x-1)$$

Q Are there values of λ for which G does not exist? (\Rightarrow homogeneous problem has a non-trivial solution)

G fails to exist if $B_1 u_2 = 0$ or $B_2 u_1 = 0$

$$B_1 u_2 = 0 \Rightarrow \sin 2(-1) - 2 \cos 2(-1) = 0$$

$$B_1 u = u(0) = 0 \quad -(\sin 2 + 2 \cos 2) = 0$$

solutions of this equation give values for λ for which G does not exist