

9/13/2017

Recall

$$\text{if } B_1 u_2 = 0 \Rightarrow$$

$$-(hu_2 + 2\cos 2) = 0$$

$$\underline{\text{Note: }} B_2 u_1 = 0$$

$$u_1 = hu_2 x$$

$$B_2 u = u(1) + u'(1) \Rightarrow$$

$$hu_2 + 2\cos 2 = 0$$

This equation has  $\infty$  many roots  $2$ . For these values of  $2$ , there is no  $G(x, \xi)$ , i.e. Green's function does not exist.

For  $G$  to exist, we need to require

$$hu_2 + 2\cos 2 \neq 0$$

BCs are separated  $\Rightarrow$

$$G(x, \xi) = \begin{cases} A u_1(x) u_2(\xi) & 0 \leq x < \xi \leq 1 \\ A u_1(\xi) u_2(x) & 0 \leq \xi < x \leq 1 \end{cases}$$

This satisfies all conditions for  $G$  except of the jump of  $\left[ \frac{dG}{dx} \right]_{x=\xi^-}^{\xi^+}$  which gives  $A$ .

$$\frac{d^2G}{dx^2} + \lambda^2 G = \delta(x - \xi) \quad | \quad \int_{\xi^-}^{\xi^+} \cdot dx$$

$$a_0(x) = 1$$

$$\Rightarrow \left[ \frac{dG}{dx} \right]_{x=\xi^-}^{x=\xi^+} = 1$$

$$A = \frac{1}{a_0(\xi) W(\xi)}$$

$$\therefore G(x, \xi) = \begin{cases} \frac{u_1(x) u_2(\xi)}{a_0(\xi) \bar{W}(\xi)} & 0 \leq x < \xi \leq 1 \\ \frac{u_1(\xi) u_2(x)}{a_0(\xi) \bar{W}(\xi)} & 0 \leq \xi < x \leq 1 \end{cases}$$

$$W(\xi) = 0 \quad \Leftrightarrow \quad B_1 u_2 = 0 \quad \text{or} \quad B_2 u_1 = 0$$

$$A = \frac{1}{a_0(\xi) W(\xi)} \Rightarrow A a_0(\xi) \bar{W}(\xi) = 1$$

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1(x) u_2'(x) - u_2(x) u_1'(x)$$

$$W(u_1, u_2)(\xi) = u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi) \Leftrightarrow$$

$$u_1(x) = \sin 2x$$

$$u_2(x) = \sin 2(x-1) - 2 \cos 2(x-1)$$

$$\Rightarrow \sin 2x \left[ 2 \underline{\cos 2(x-1)} + 2^2 \underline{\sin 2(x-1)} \right] -$$

$$- \left[ \underline{\sin 2(x-1)} - 2 \underline{\cos 2(x-1)} \right] 2 \cos 2x =$$

$$= 2 \left[ \sin 2x \cdot \cos 2(x-1) - \cos 2x \cdot \sin 2(x-1) \right] +$$

$$+ 2^2 \left\{ \sin 2x \cdot \sin 2(x-1) + \cos 2x \cdot \cos 2(x-1) \right\} \quad \square$$

$$\sin a \cdot \cos b \mp \cos a \cdot \sin b = \sin(a \mp b)$$

$$\cos a \cdot \cos b \pm \sin a \cdot \sin b = \cos(a \mp b)$$

$$\begin{aligned} & \square 2 \cdot \sin(2x - 2(x-1)) + 2^2 \underbrace{\cos(2x - 2(x-1))}_{\cos(-2) = \cos 2} = \\ & = 2 \sin 2 + 2^2 \cos 2 \end{aligned}$$

$$\therefore \boxed{W(u_1, u_2)(x) = 2 \sin 2 + 2^2 \cos 2}$$

is independent of  $x$

Hence,  $\bar{W} = \bar{W}(0)$  or  $\bar{W} = \bar{W}(1)$

$$\bar{W} = \bar{W}(0) = u_1(0) u_2'(0) - u_2(0) u_1'(0) \quad \textcircled{=} \quad \text{with } u_1(0) = 0$$

$$B_1 u_1(0) = 0 \Rightarrow u_1(0) = 0$$

$$B_2 u_2(1) = 0 \Rightarrow u_2(1) + u_2'(1) = 0$$

$$\textcircled{=} -u_2(0) u_1'(0)$$

$$\Rightarrow u_2(0) = 0$$

Now, if we require  $B_1 u_2(0) = 0$

$$\text{or } B_2 u_1(1) = 0$$

Then we get  $\bar{W} = \bar{W}(0) = 0$

Similarly, one can consider  $\bar{W} = \bar{W}(1)$  to show

$\bar{W} = 0$  if we impose  $B_1 u_2 = 0$  or  $B_2 u_1 = 0$

In summary, if we require  $\bar{W}(u_1, u_2)(\xi) = 0$  for the Green's function to exist, we require

$$2(\sin 2 + 2 \cos 2) \neq 0$$

Then

$$G(x, \xi) = \begin{cases} \frac{\sin x \cdot (\sin(\xi-1) - 2\cos(\xi-1))}{2(\sin 2 + 2\cos 2)} & 0 \leq x < \xi \leq 1 \\ \frac{\sin \xi \cdot (\sin(x-1) - 2\cos(x-1))}{2(\sin 2 + 2\cos 2)} & 0 \leq \xi < x \leq 1 \end{cases}$$

Note Green's function has simple poles, roots of denominator are simple.

Since the BCs in BVP (3.10) for  $u$  are homogeneous, the solution of this BVP is

$$u(x) = \int_0^1 G(x, \xi) f(\xi) d\xi$$

Q How do we solve for  $u$  if BCs are inhomogeneous, ie.  $B_i u = g_i$ ,  $i=1, \dots, n$ ?

We need some theoretical results.

### 3.4 Adjoint Green's function

$$G^*(x, \xi) = G(\xi, x)$$

Def For the BVP (3.1)  $Lu=f$   $x \in (a, b)$   
 $B_i u = c_i$   $i=1, \dots, n$

with general inhomogeneous data  $\{f, c_i\}$ ,  
 the (direct) Green's function  $G(x, \xi)$  is  
 the solution of

$$(3.2) \quad \begin{aligned} L G(x, \xi) &= \delta(x - \xi) & x, \xi &\in (a, b) \\ B_i G &= 0 & i &= 1, \dots, n \end{aligned}$$

Def The adjoint Green's function  $G^*(x, \xi)$   
 is the solution of

$$L^* G^*(x, \xi) = \delta(x - \xi) \quad x, \xi \in (a, b)$$

$$B_i^* G^*(x, \xi) = 0 \quad i = 1, \dots, n$$

$$\langle \cdot, \cdot \rangle = \int_a^b \cdot \cdot \cdot dx$$

Basic result:

$$G^*(x, \xi) = G(\xi, x)$$

so we need only construct one of  $G$  or  $G^*$  to get both.

Proof Use the Lagrange identity

$$\langle v, Lu \rangle = [J(u, b)]_a^b + \langle u, L^*v \rangle$$

with  $v = G^*(x, \xi)$ ,  $u = G(x, \xi)$ .

Derivatives in  $L$  and integration are wrt  $x$ .

$$\begin{aligned} \langle G^*(x, \gamma), LG(x, \xi) \rangle &= [J(G, G^*)]_{x=a}^{x=b} + \\ &\quad \underbrace{\delta(x-\xi)}_{\delta(x-\gamma)} \\ &\quad + \langle G(x, \xi), L^*G^*(x, \gamma) \rangle \end{aligned}$$

$$\int_a^b G^*(x, \gamma) \delta(x-\xi) dx = [J(G, G^*)]_{x=a}^{x=b} +$$

$$\left. \begin{array}{l} B_i G = 0 \\ B_i^* G = 0 \end{array} \right\} \Rightarrow [J]_a^b = 0$$

$$+ \int_a^b G(x, \xi) \delta(x-\gamma) dx$$

$$G^*(\xi, \eta) = G(\eta, \xi)$$

Now, relabel  $\eta$  as  $x \Rightarrow G^*(\xi, x) = G(x, \xi)$

COROLLARY: if  $L = L^*$ , i.e. the operator  $L$  of BVP is self-adjoint, then the Green's function is symmetric, i.e.

$$G(x, \xi) = G(\xi, x)$$