2/02/2018

Solution $u_I(x)$ to the problem (1.2.2)

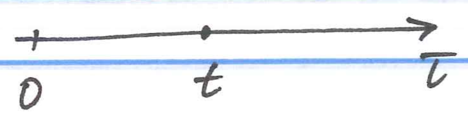
$$\begin{cases} u_t - u_{xx} = p(x,t), & -\infty < x < \infty, \quad t > 0 \\ u(x, 0^-) = 0 \end{cases}$$

is

$$u_I(x,t) = \int_0^\infty \int_{-\infty}^\infty p(\xi,\tau)\, F(x-\xi, t-\tau)\, d\xi\, d\tau \; \textcircled{=}$$

here x and t are fixed.

$0 < t < \infty$



$$\int_0^\infty \dots d\tau = \int_0^t \dots d\tau + \int_t^\infty \dots d\tau$$

Recall

$$F(x-\xi, t-\tau) = H(t-\tau) \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}$$

$$H(t-\tau) = \begin{cases} 1, & t-\tau > 0 \\ 0, & t-\tau < 0 \end{cases} \quad \text{or} \quad \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}$$

$$\textcircled{=} \int_0^t \int_{-\infty}^\infty p(\xi,\tau)\, F(x-\xi, t-\tau)\, d\xi\, d\tau$$

since $F(x-\xi, t-\tau)=0$ for $\tau > t$

$$\underbrace{\int_0^t \dots d\tau}_{0<\tau<t} + \underbrace{\int_t^\infty \dots d\tau}_{\tau>t}$$

Back to u_{II}. Last time we showed that

$$u_{II}(x,t) = \int_0^t \int_{-\infty}^{\infty} f(\xi)\,\delta(\tau)\,\frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}\,d\xi\,d\tau =$$

$$0<\tau<t, \quad -\infty<x<\infty$$

since $\int g(\tau)\,\delta(\tau-\tau_0)\,d\tau = g(\tau_0)$

(with $\tau_0=0$)

$$= \int_{-\infty}^{\infty} f(\xi)\,\frac{e^{-\frac{(x-\xi)^2}{4t}}}{2\sqrt{\pi t}}\,d\xi$$

ie.

$$u_{II}(x,t) = \int_{-\infty}^{\infty} f(\xi)\,\frac{e^{-\frac{(x-\xi)^2}{4t}}}{2\sqrt{\pi t}}\,d\xi$$

or

(1.2.5) $$\boxed{u_{II}(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\xi)\,e^{-\frac{(x-\xi)^2}{4t}}\,d\xi}$$

Hence, solution of the entire problem (1.2.1) is

(1.2.6)
$$\boxed{u(x,t) = \int_0^t \int_{-\infty}^{\infty} p(\xi,\tau)\, \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}}}{2\sqrt{\pi(t-\tau)}}\, d\xi\, d\tau + \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi}$$

We can verify directly that (1.2.6) solves

$u_t - u_{xx} = p(x,t)$

Solution: HW

You would need to use here the result

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) F(x-\xi, t-\tau) = \delta(x-\xi, t-\tau)$$

Let's verify that the initial condition

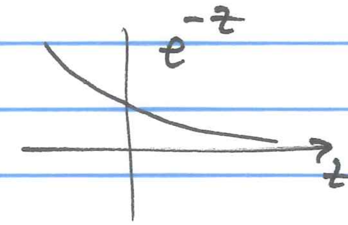
$u(x,0^+) = f(x)$ is satisfied by showing that

$\lim_{t\to 0^+} u_{II}(x,t) = f(x)$, since $u_I(x,0)=0$

by splitting assumption.

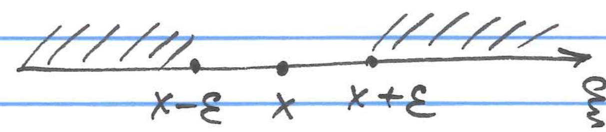
$$\underline{u_{II}}(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi$$

Note that $e^{-\frac{(x-\xi)^2}{4t}} \to 0$



uniformly as $t \to 0^+$ if x is away from ξ, i.e. on

$(-\infty, x-\varepsilon)$ and $(x+\varepsilon, +\infty)$.



Consider $\xi \in (x-\varepsilon, x+\varepsilon)$, $\varepsilon > 0$, small.

$$u(x,0^+) = \lim_{t\to 0^+} \int_{x-\varepsilon}^{x+\varepsilon} f(\xi) \frac{e^{-\frac{(x-\xi)^2}{4t}}}{2\sqrt{\pi t}}\, d\xi \;=$$

Contribution from $(\infty, x-\varepsilon) \cup (x+\varepsilon, +\infty)$ tends to 0 as $t \to 0^+$, so we can neglect it.

$=$

$$\zeta = \frac{x-\xi}{2\sqrt{t}} \qquad \xi = x-\varepsilon \Rightarrow \zeta = \frac{x-(x-\varepsilon)}{2\sqrt{t}} = \frac{\varepsilon}{2\sqrt{t}}$$

$$d\zeta = -\frac{d\xi}{2\sqrt{t}} \qquad \xi = x+\varepsilon \to \zeta = \frac{x-(x+\varepsilon)}{2\sqrt{t}} = \frac{-\varepsilon}{2\sqrt{t}}$$

$$\xi = x - 2\sqrt{t}\,\zeta$$

$$\overset{-\frac{\varepsilon}{2\sqrt{t}} \to -\infty \text{ as } t\to 0^+,\ \varepsilon>0 \text{ fixed}}{=} \lim_{t\to 0^+} \int_{\underbrace{\frac{\varepsilon}{2\sqrt{t}}}_{\to +\infty}}^{-\frac{\varepsilon}{2\sqrt{t}}} \underbrace{f(x-2\sqrt{t}\,\xi)}_{\to x} \frac{e^{-\xi^2}}{2\sqrt{\pi t}} \left(-2\sqrt{t}\, d\xi\right)$$

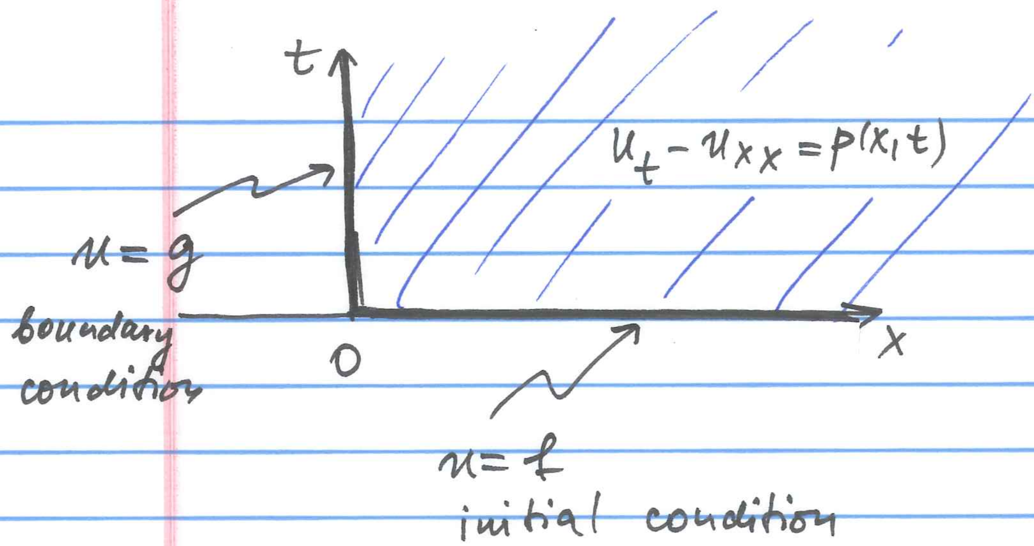
$$= -\int_{+\infty}^{-\infty} f(x) \frac{e^{-\xi^2}}{\sqrt{\pi}}\, d\xi = \frac{f(x)}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi}_{=\sqrt{\pi}} =$$

$$= f(x) \quad \checkmark$$

1.3 SEMI-INFINITE DOMAIN INITIAL BOUNDARY VALUE PROBLEM

I PROBLEM OF 1st KIND

$$(1.3.1)\quad \begin{cases} u_t - u_{xx} = p(x,t), & x \in (0,+\infty),\quad t>0 \\ \text{IC} \quad u(x,0) = f(x) \\ \text{BC} \quad u(0,t) = g(t) \end{cases}$$



Note that equation, initial and boundary conditions are inhomogeneous due to the presence of p, f, and g.

1st kind: BC on u

2nd kind: BC on u_x

Because of linearity, we can use superposition and write solution of (1.3.1) as

$$u = u_{I} + u_{II} + u_{III}$$

where

(1.3.2) u_{I} satisfies $u_t - u_{xx} = p$, $u(x,0) = 0$, $u(0,t) = 0$

(1.3.3) u_{II} satisfies $u_t - u_{xx} = 0$, $u(x,0) = f$, $u(0,t) = 0$

(1.3.4) u_{III} satisfies $u_t - u_{xx} = 0$, $u(x,0) = 0$, $u(0,t) = g$

For u_I and u_{II}, problems are similar to those for infinite domain (section 1.2). But we have to account for semi-infinite domain. We will use the METHOD OF IMAGES.