2/05/2018

The GREEN's FUNCTION (OF THE 1^{st} KIND) is a solution of

$$
(1.3.5)\quad \begin{cases} u_t - u_{xx} = \delta(x-\xi)\,\delta(t-\tau), & x, \xi, t, \tau \in (0, +\infty) \\ u(x, \tau^-) = 0 \\ u(0, \tau) = 0 \end{cases}
$$

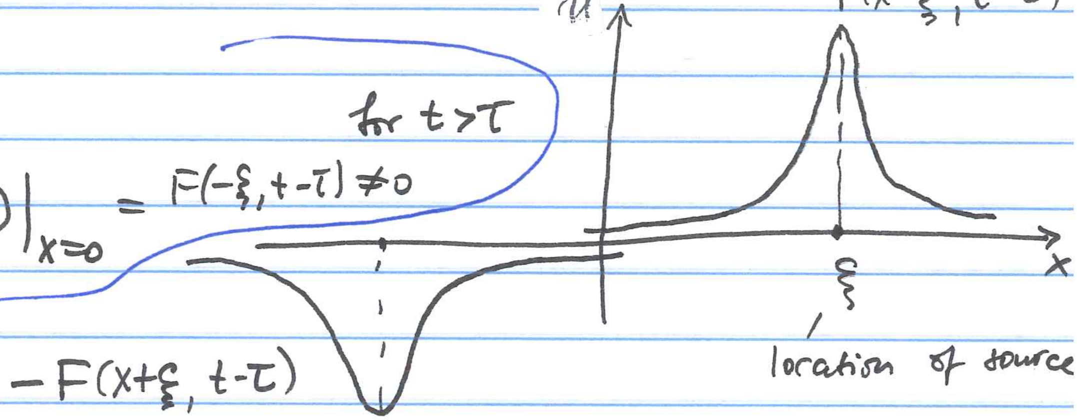
We think about ξ and τ to fixed.
We have homogeneous data (IC & BC) and δ-function source in the field equation.

We denote this Green's function as $G_I(x, \xi; t, \tau)$ and will use the fundamental solution $F(x, \xi; t, \tau)$ to construct $G_I(x, \xi; t, \tau)$.

$F(x-\xi, t-\tau)$ satisfies PDE and the IC (because it is multiplied by Heaviside function) but it does not satisfy BC.

Note that

$$F(x-\xi, t-\tau)\Big|_{x=0} = F(-\xi, t-\tau) \neq 0 \quad \text{for } t > \tau$$



We introduce an image (of the original source) given by reflection in the boundary $x=0$, i.e. at $x=-\xi_1$ $t=\tau$ with "strength" -1, i.e. solution of

$$\begin{cases} u_t - u_{xx} = -\delta(\overset{x-(-\xi)}{\overset{\downarrow}{x+\xi}})\,\delta(t-\tau) \\ u(x,\tau^-)=0 \end{cases}$$

which is $-F(x+\xi_1, t-\tau)$.

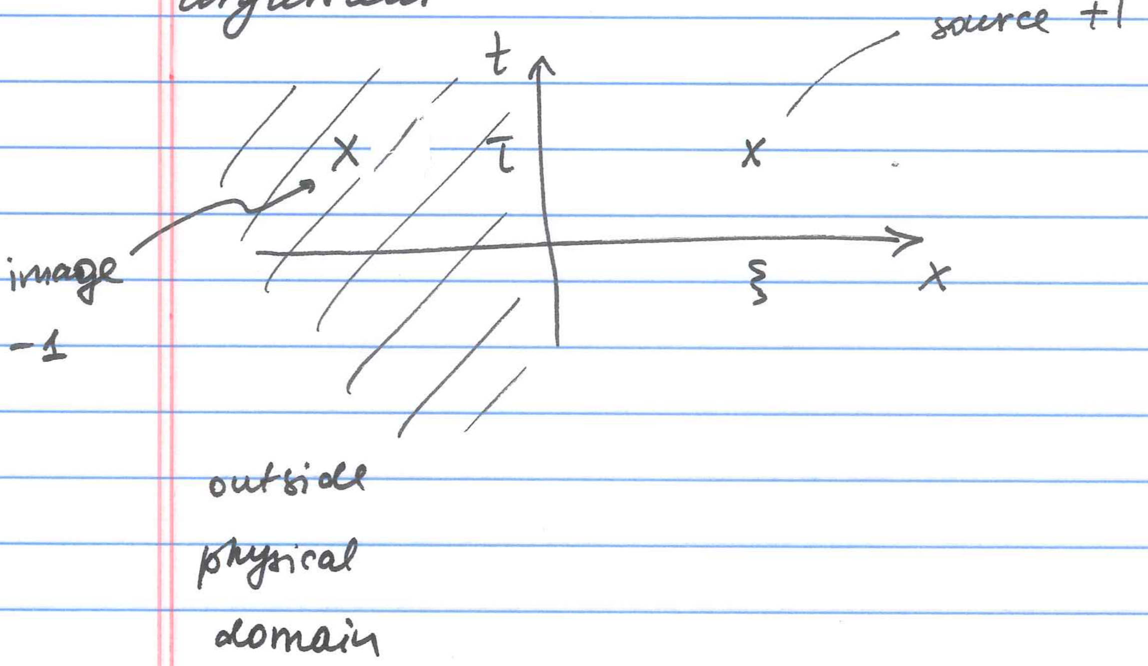
Now consider the response due to the source and its image

$$F(x-\xi_1, t-\tau) - F(x+\xi_1, t-\tau)$$

This satisfies the PDE since the image is outside the physical/computational domain $x>0$, $t>0$. It satisfies the IC and (now) satisfies the BC at $x=0$ since

at $x=0$: $\quad F(-\xi_1, t-\tau) - F(\xi_1, t-\tau) = 0$

because F is symmetric function in 1st argument



Hence,

$$G_I(x,\xi;t,\tau) = F(x-\xi, t-\tau) - F(x+\xi, t-\tau) =$$

$$= \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}}\left(e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}\right)$$

Then, reasoning similarly as in Section 1.2, we can write

$$\boxed{\underset{\text{solution of (1.3.2)}}{u_I(x,t)} = \int_0^t \int_0^\infty p(\xi,\tau)\, G_I(x,\xi;t,\tau)\, d\xi\, d\tau} \tag{1.3.6}$$

Note that u_I is the integral of Green's function multiplied by $p(x,\tau)$ over the physical domain.

u_{II} Similarly to Section 1.2, the problem (1.3.3) for $u_{II}(x,t)$ is equivalent to

$$\begin{cases} u_t - u_{xx} = f(x)\,\delta(t) & \\ u(x,0^-) = 0 & x,t \in (0,+\infty) \\ u(0,t) = 0 & \end{cases}$$

For this problem $p(x,t) = f(x)\,\delta(t)$. Hence,

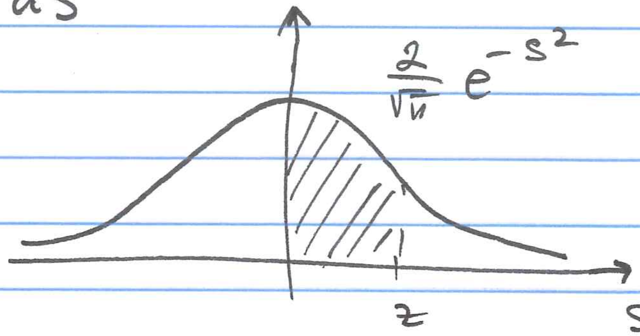
$$u_{II}(x,t) = \int_0^t \int_0^\infty f(\xi)\,\underset{\substack{\uparrow \\ \tau - 0}}{\delta(\tau)}\, G_I(x,\xi;t,\tau)\,d\xi\,d\tau =$$

(1.3.7) $$= \boxed{\int_0^\infty f(\xi)\, G_I(x,\xi;t,0)\,d\xi = u_{II}(x,t)}$$

Ex Consider the case when $f(x) = C = \text{const}$

We can show that in this case the integral (1.3.7) simplifies to give $u(x,t)$ in terms of the error function

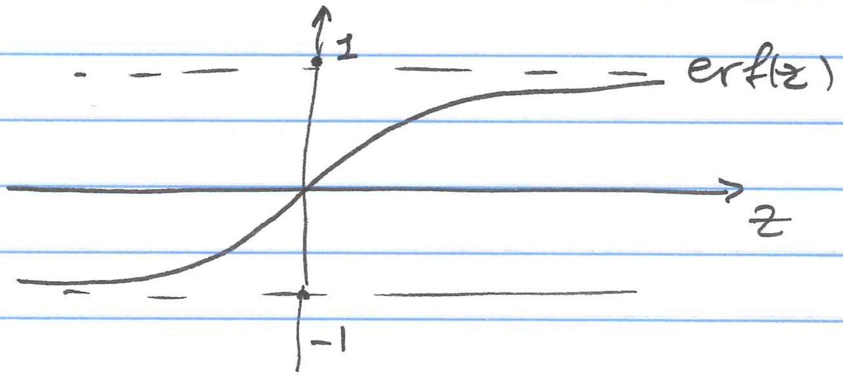
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds$$



This function is normalized:

$$\operatorname{erf}(\infty) = 1$$

$\operatorname{erf}(z)$ is an odd function



$$\underset{u_{II}}{\underset{\parallel}{u(x,t)}} = \int_0^\infty \underbrace{f(\xi)}_{\overset{\parallel}{C}} G_I(x,\xi;t,0)\, d\xi =$$

$$= \frac{C}{2\sqrt{\pi t}} \left(\int_0^\infty \underbrace{e^{-\frac{(x-\xi)^2}{4t}}}_{Ⓘ} d\xi - \int_0^\infty \underbrace{e^{-\frac{(x+\xi)^2}{4t}}}_{Ⓘⓘ} d\xi \right)$$

$$\textcircled{I} = \int_0^{\infty} e^{-\frac{(x-\xi)^2}{4t}}\, d\xi = \left| \begin{array}{l} s = \frac{x-\xi}{2\sqrt{t}} \\ ds = \frac{-d\xi}{2\sqrt{t}} \\ \xi = 0 \Rightarrow s = \frac{x}{2\sqrt{t}} \\ \xi \to \infty \Rightarrow x \to -\infty \end{array} \right| =$$

$$= \int_{\frac{x}{2\sqrt{t}}}^{-\infty} e^{-s^2} (-2\sqrt{t})\, ds$$

$$\textcircled{II} = \int_0^{\infty} e^{-\frac{(x+\xi)^2}{4t}}\, d\xi = \left| \begin{array}{l} s = \frac{x+\xi}{2\sqrt{t}} \\ ds = \frac{d\xi}{2\sqrt{t}} \\ \xi = 0 \Rightarrow s = \frac{x}{2\sqrt{t}} \\ \xi \to \infty \Rightarrow s \to \infty \end{array} \right| =$$

$$= \int_{\frac{x}{2\sqrt{t}}}^{+\infty} e^{-s^2}\, 2\sqrt{t}\, ds$$

$$\boxed{=}\ \frac{c}{2\sqrt{\pi t}}\left(\textcircled{I} - \textcircled{II}\right) = \frac{c}{2\sqrt{\pi t}}\left(\int_{\frac{x}{2\sqrt{t}}}^{-\infty} e^{-s^2}(-2\sqrt{t})\,ds\right.$$

$$\left.-\int_{\frac{x}{2\sqrt{t}}}^{+\infty} e^{-s^2}\,2\sqrt{t}\,ds\right) =$$

$$= \frac{c}{\sqrt{\pi}}\left(\underbrace{\int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-s^2}\,ds}_{ⓐ} - \underbrace{\int_{\frac{x}{2\sqrt{t}}}^{+\infty} e^{-s^2}\,ds}_{ⓑ}\right)$$

$z = \frac{x}{2\sqrt{t}}$

$$ⓐ = \int_{-\infty}^{z} \dots = \int_{-\infty}^{0} \dots + \int_{0}^{z} \dots \overset{e^{-s^2}}{\underset{\text{even}}{=}} \int_{0}^{\infty} \dots + \int_{0}^{z} \dots$$

$$ⓑ = \int_{z}^{\infty} \dots = \int_{0}^{\infty} \dots - \int_{0}^{z} \dots$$