2/7/2018

#3
HW#1

$Lu = f(t), \quad t \geq 0$

$u(0) = 0, \quad u'(0) = 0$

$$u(t) = \int_0^t G(t,\tau) f(\tau) d\tau$$

L : 2^{nd} order diff. operator

$$L = a_2 \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_0, \quad a_2 \neq 0$$

or $$L = \frac{d^2}{dt^2} + a \frac{d}{dt} + b$$

$$Lu = f \quad \text{or} \quad u'' + au' + bu = f \qquad (1)$$

$$u'(t) = G(t,t) f(t) + \int_0^t G_t \cdot f \, d\tau$$

$$u'' = \left(G f(t)\right)' + G_t \cdot f + \int_0^t G_{tt} \cdot f \, d\tau$$

$$\frac{d}{dt} \int_{a_0}^t f(\tau) d\tau = f(t)$$

Substitute into (1)

$$f(t) \overset{?}{=} \int_0^t \left(G_{tt} f + a\, G_t \cdot f + b\, G \cdot f\right) d\tau +$$

$$+ \underline{G(t,t) f(t) + \left(f(t) G(t,t)\right)' + G_t(t,t) \cdot f(t)}$$

$G(t,\tau)$ satisfies

$$LG = \delta(t) \qquad \Rightarrow \quad G_{tt} + aG_t + bG = \delta(t)$$

$$G(0)=0, \quad G'(0)=0$$

$$\Rightarrow \int_0^t \underbrace{\left(G_{tt} + aG_t + bG\right)}_{\delta(t-0)} f(\tau)\,d\tau = f(t)$$

Note:

$$u'(t) = G(t,t)\,f(t) + \int_0^t G_t \cdot f(\tau)\,d\tau$$

$$u'(0)=0 \Rightarrow \left(G(t,t)\,f(t)\right)\Big|_{t=0} + \cancel{\int_0^0 \dots d\tau} \to 0$$

$$\Rightarrow \left(G(t,t)\,f(t)\right)\Big|_{t=0} = 0 \qquad (*)$$

OR, try relating $\underbrace{\qquad}$ terms to an ODE w/ IC (*).

HW #1

#1.2.1

$$2x u_t - u_{xx} = 0 \qquad 0 \le x < \infty, \ t \geq 0$$

BCs

$$u(0,t) = C_1 = \text{const} \qquad t > 0$$

$$u(\infty,t) = C_2 = \text{const} \qquad t > 0$$

IC

$$u(x,0) = C_3 = \text{const}$$

Let $u(x,t) = \alpha F(\beta x, \gamma t)$ $\qquad \bar{x} = \beta x, \ \bar{t} = \gamma t$

$$u_t = \alpha F_{\bar{t}} \cdot \gamma \qquad u_x = \alpha F_{\bar{x}} \cdot \beta \qquad u_{xx} = \alpha \beta^2 F_{\bar{x}\bar{x}}$$

$$2 \frac{\bar{x}}{\beta} \cdot \alpha \gamma F_{\bar{t}} - \alpha \beta^2 F_{\bar{x}\bar{x}} = 0$$

$$2 \frac{\alpha \gamma}{\beta} \bar{x} F_{\bar{t}} - \alpha \beta^2 F_{\bar{x}\bar{x}} = 0 \quad \Big| \ \frac{\beta}{\alpha\gamma}$$

$$2 \bar{x} F_{\bar{t}} - \frac{\cancel{\alpha} \beta^2 \cdot \beta}{\cancel{\alpha} \gamma} F_{\bar{x}\bar{x}} = 0$$

$$2 \bar{x} F_{\bar{t}} - \frac{\beta^3}{\gamma} F_{\bar{x}\bar{x}} = 0$$

$F(\bar{t}, \bar{x})$ is also a solution $\Rightarrow$ $\frac{\beta^3}{\gamma} = 1$

$u(0,t)=C_1$

$\alpha F(0,\bar{t})=C_1 \Rightarrow \quad \alpha=1$

$u(\infty,t)=C_2 \Rightarrow \alpha F(\infty,\bar{t})=C_2 \Rightarrow \alpha=1$

$u(x,0)=C_3 \Rightarrow \alpha F(\bar{x},0)=C_3 \Rightarrow \alpha=1$

$\Rightarrow \boxed{\gamma=\beta^3, \quad \alpha=1}$

$\Rightarrow u(x,t)=F(\beta x, \beta^3 t)$

$$\Rightarrow \frac{(\beta x)^3}{\beta^3 t}=\frac{\bar{x}^3}{\bar{t}}=\frac{x^3}{t}$$

Let $\boxed{\zeta=\frac{x^3}{t}} \Rightarrow \boxed{u(x,t)=f(\zeta)}$

$u_t=f'\cdot\left(-\frac{x^3}{t^2}\right) \qquad u_x=f'\cdot\frac{3x^2}{t}=\frac{3}{t}\left(f'\cdot x^2\right)$

$$u_{xx}=\frac{3}{t}\left(f''\cdot\frac{3x^2}{t}\cdot x^2+f'\cdot 2x\right)$$

$$2x\cdot\left(-\frac{x^3}{t^2}\right)f'-\frac{3}{t}\left(f''\frac{3x^4}{t}+f'\cdot 2x\right)=0$$

$$-\frac{2x^4}{t^2}f'-\frac{3x^4}{t^2}f''-\frac{6x}{t}f'=0 \quad \Big|\cdot\frac{t}{x}$$

$$-\frac{2x^3}{t}f'-3\frac{x^3}{t}f''-6f'=0 \quad \Big|\cdot(-1)$$

$$2\xi f' + 3\xi f'' + 6f' = 0$$

$$(2\xi + 6) f' + 3\xi f'' = 0$$

Let $g = f'$. Then

$$(2\xi + 6) g + 3\xi g' = 0 \qquad \text{separable ODE}$$

$$(2\xi + 6) g = -3\xi \frac{dg}{d\xi}$$

$$\frac{2\xi + 6}{3\xi} d\xi = -\frac{dg}{g}$$

$$-\left(\frac{2}{3} + \frac{2}{\xi}\right) d\xi = +\frac{dg}{g}$$

$$\ln|g| = -\left(\frac{2}{3}\xi + 2\ln|\xi|\right) + \tilde{C}$$

$$g(\xi) = C_1 e^{-\left(\frac{2}{3}\xi + 2\ln|\xi|\right)} = C_1 e^{-\frac{2}{3}\xi} \cdot \xi^2$$

$$f(\xi) = C_1 \int e^{-\frac{2}{3}\xi} \cdot \xi^2 \, d\xi$$

Ex $f(x) = c = \text{const}$ (Cont'd)

From last lecture:

$$u(x,t) = \int_0^{\infty} \underbrace{f(\xi)}_{c} \, G_I(x,\xi;t,\tau)\, d\xi = \dots$$

$$= \frac{c}{\sqrt{\pi}} \Bigg(\underbrace{\int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds}_{ⓐ} - \underbrace{\int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-s^2} ds}_{ⓑ} \Bigg) \; ⊜$$

$z = \dfrac{x}{2\sqrt{t}}$

$$(a) = \int_{-\infty}^{z} \dots = \int_{-\infty}^{0} \dots + \int_0^z \underset{\text{is even}}{\overset{e^{-s^2}}{=}} \int_0^{\infty} + \int_0^z$$

$$(b) = \int_z^{\infty} \dots = \int_0^{\infty} \dots - \int_0^z \dots$$

$$⊜ \; \frac{c}{\sqrt{\pi}} \Bigg[\underbrace{\cancel{\int_0^{\infty} e^{-s^2} ds}}_{\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}\operatorname{erf}(\infty)} + \int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds -$$

$$- \Bigg(\underbrace{\cancel{\int_0^{\infty} e^{-s^2} ds}}_{\sqrt{\pi}/2} - \int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds \Bigg) \Bigg] = \frac{c}{\sqrt{\pi}} \cdot 2 \int_0^{\frac{x}{2\sqrt{t}}} e^{-s^2} ds$$

$$= c \cdot \text{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

$$\therefore \boxed{u(x,t) = c \cdot \text{erf}\left(\frac{x}{2\sqrt{t}}\right)} \quad \text{if} \quad f(x) = c$$

<u>Note</u> $u(x,t) = \text{const}$ on a path $x = \text{const} \cdot \sqrt{t}$

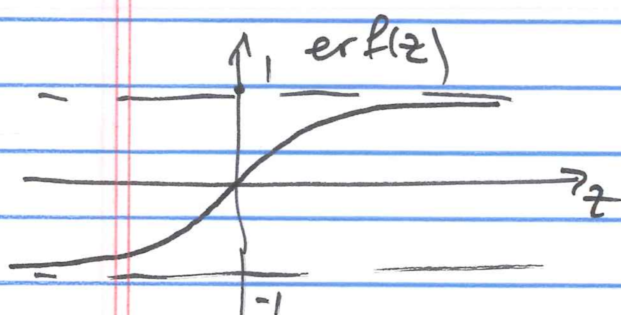
(x is proportional to $\sqrt{t}$)

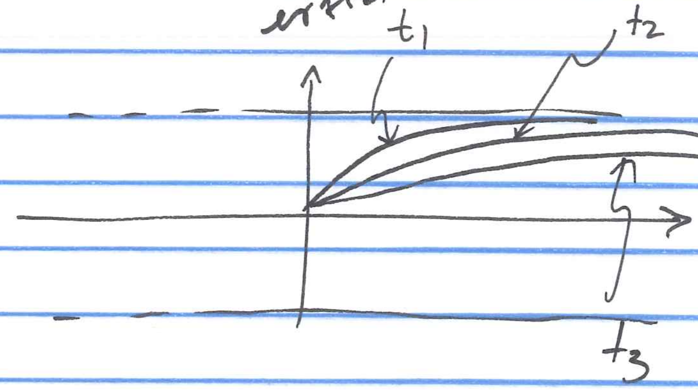
We can verify that the above $u(x,t)$ satisfies PDE, i.e.

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) = 0$$

$$\lim_{t \to 0^+} u(x,t) = c \cdot \text{erf}(\infty) = c \cdot 1 = c = f(x)$$

Similarly, $\lim_{x \to 0} u(x,t) = 0$





$t_1 < t_2 < t_3$

(For greater times)
As time increases, the graph of u is more shallow.

u_{III}

$$u_t - u_{xx} = 0 \qquad x \in (0, +\infty),\ t > 0$$

$$u(x,0) = 0 \qquad u(0,t) = g(t)$$

We introduce a "homogenizing transformation" for a new function $w(x,t)$ such that $w(0,t) = 0$. The new problem for w

will be inhomogeneous in PDE and IC, but homogeneous in BC. We know how to solve problem w/ inhomog. PDE (problem for u_I) and problem w/ inhomog. IC (problem for u_{II}).

Homogenizing transformation

Let $w(x,t) = u(x,t) - g(t)$: difference between temperature u at a field point (x,t) and temperature at the boundary

Problem for w.

$w_t = u_t - \dot{g} \qquad w_x = u_x, \qquad w_{xx} = u_{xx}$

$$w_t - w_{xx} = u_t - \dot{g} - u_{xx} = \underbrace{u_t - u_{xx}}_{=0} - \dot{g} = -\dot{g}(t)$$

$$\Rightarrow \boxed{w_t - w_{xx} = -\dot{g}(t)}$$

$$\underbrace{w(x,0^+)}_{\lim_{t\to 0^+} w(x,t)} = \underbrace{u(x,0^+)}_{0} - g(0^+) = -g(0^+)$$

$$\Rightarrow \boxed{w(x,0^+) = -g(0^+)}$$

$$w(0,t) = u(0,t) - g(t) = g(t) - g(t) = 0$$

by construction

$$\Rightarrow \boxed{w(0,t) = 0}$$

Hence, problem for $w(x,t)$ is

$$\begin{cases} w_t - w_{xx} = -\dot{g}(t) \\ w(x,0^+) = -g(0^+) \\ w(0,t) = 0 \end{cases}$$

This problem is equivalent to

$$\begin{cases} w_t - w_{xx} = -\dot{g}(t) - g(0^+)\,\delta(t), & x \in (0,\infty) \\ & t > 0 \\ w(x,0^-) = 0 & \\ w(0,t) = 0 & \end{cases}$$

Let's prove this.

$$w(x,0^+) = \lim_{t \to 0^+} w(x,t) = \int_{0^-}^{0^+} w_t(x,t)\,dt =$$

$$= \int_{0^-}^{0^+} \left[w_{xx} - \dot{g}(t) - g(0^+)\,\delta(t) \right] dt =$$

smooth

→ contribution

$$= -g(0^+) \underbrace{\int_{0^-}^{0^+} \delta(t)\,dt}_{=1} = -g(0^+)$$